FALL 2003 FINAL EXAM FOR SM261
0755 DECEMBER 11, 2003

SHOW ALL WORK

PART 1. On this part you may **NOT** use a calculator. Show all your work.

1. Complete the following definitions.

a. A subset W of $\mathbb{R}^n$ is called a subspace of $\mathbb{R}^n$ if...
Answer: *it is closed under vector addition and scalar multiplication.*
b. If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are vectors in $\mathbb{R}^n$, then span$(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m)$=...
Answer: *the subspace of all linear combinations of the form*

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m,$$

where $c_i \in \mathbb{R}$.
c. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then the image of T is...
Answer: *the set of all $y \in \mathbb{R}^m$ for which there exists $x \in \mathbb{R}^n$ such that $T(x) = y$, i.e.,*

$$\{y \in \mathbb{R}^m \mid \text{there exists } x \in \mathbb{R}^n \text{ such that } T(x) = y\},$$

in set-theoretic notation.
d. A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ in a subspace V of $\mathbb{R}^n$ forms a basis for V if...
Answer: *they are linearly independent vectors which span V.*
e. If V is a subspace of $\mathbb{R}^n$, then the dimension of V is...
Answer: *the number of elements in a basis for V (necessarily $\leq n$).*

2. Consider the linear system of equations

$$\begin{aligned} x + y + 5z &= 9 \\ -x + 2y + z &= 6 \\ 3x - y + 7z &= 7 \end{aligned}$$

a. If we write the system in the matrix form $A\vec{v} = \vec{b}$, what are A, $\vec{v}$ and $\vec{b}$?

Answer:

$$A = \begin{pmatrix} 1 & 1 & 5 \\ -1 & 2 & 1 \\ 3 & -1 & 7 \end{pmatrix}, \qquad \vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} 9 \\ 6 \\ 7 \end{pmatrix}.$$

b. Find the reduced row echelon form of the augmented matrix $[A|\,\vec{b}\,]$ for this system. Show all steps.

Answer:

$$\operatorname{rref}\begin{pmatrix} 1 & 1 & 5 & 9 \\ -1 & 2 & 1 & 6 \\ 3 & -1 & 7 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

c. Solve the system.

Answer: $x + 3z = 4$, $y + 2z = 5$, *so*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 - 3z \\ 5 - 2z \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} z.$$

The solution set is the line in $\mathbb{R}^3$ *consisting of all points* (x, y, z) *as above.*

d. Is the solution set a subspace of $\mathbb{R}^3$? Explain. If so, what is its dimension?

Answer: *No.*

e. Is the solution set of the corresponding homogenous system $Av = 0$ a subspace of $\mathbb{R}^3$? Explain. If so, what is its dimension?

Answer: *Yes. It is the line in* $\mathbb{R}^3$ *through the origin consisting of all points* (x, y, z) *such that*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} z.$$

This has dimension 1.

3. Find the determinant of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 3 & 1 & 2 & 2 \\ 4 & 0 & 0 & 2 \\ 3 & 4 & 5 & 0 \end{bmatrix}.$$

Answer: -30.

4. Find the inverse of the matrix

$$\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}.$$

Answer: $\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$.

5. Find the product AB, where

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 1 \\ 2 & -1 \\ 3 & 4 \end{bmatrix}.$$

Answer: $\begin{pmatrix} 17 & 6 \\ -1 & -5 \end{pmatrix}$.

This is the end of Part 1. Turn in your answers and proceed to Part 2.

PART 2. You may use a calculator on this part. Show all your work.

6. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation whose matrix is

$$\begin{bmatrix} 1 & 3 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

and $S : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation whose matrix is

$$\begin{bmatrix} 2 & 0 \\ 4 & 1 \\ 0 & 3 \end{bmatrix}.$$

Find the matrix of the linear transformation $S \circ T$. (Recall that $S \circ T(v) = S(T(v))$.)

Answer: $\begin{pmatrix} 2 & 6 & 0 \\ 8 & 12 & 1 \\ 12 & 0 & 3 \end{pmatrix}$

7. $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation such that $T\left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right)$, $T\left(\begin{smallmatrix} 3 \\ 1 \\ 2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ and $T\left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -3 \\ -2 \end{smallmatrix}\right)$. Find the matrix of T.

Answer: *Because of the linearity of T, we have*

$$T\left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right) = T\left(\begin{smallmatrix} 3 \\ 1 \\ 2 \end{smallmatrix}\right) - 3T\left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right) - 2T\left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) - 3\left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right) - 2\left(\begin{smallmatrix} -3 \\ -2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ -4 \end{smallmatrix}\right).$$

Therefore, the matrix of T with respect to the standard basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ is

$$[T(e_1), T(e_2), T(e_3)] = \begin{pmatrix} 2 & 1 & -3 \\ 3 & -4 & -2 \end{pmatrix}.$$

8. Let A be the matrix

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

a. Find a basis for $\ker(A)$.

Answer: *Note that*

$$\operatorname{rref}(A, \vec{0}) = \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, we can compute the kernel of A in $\mathbb{R}^6$ using the equations. $x_1 - x_3 + x_5 - 2x_6 = 0$, $x_2 - 2x_3 + x_6 = 0$, $x_4 + 2x_5 - 3x_6 = 0$. These 3 equations in 6 unknowns, leave 3 free variables; let these be x_3, x_5, x_6. The kernel is the set of all $(x_1, \ldots, x_6)$ such that

$$\begin{pmatrix} x_1 \\ \vdots \\ x_6 \end{pmatrix} = \begin{pmatrix} x_3 - x_5 + 2x_6 \\ 2x_3 - x_6 \\ x_3 \\ -2x_5 + 3x_6 \\ x_5 \\ x_6 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} 2 \\ -1 \\ 0 \\ 3 \\ 0 \\ 1 \end{pmatrix}.$$

This has dimension 3.

b. Find $\dim(\ker(A))$.

c. Find a basis for $\operatorname{im}(A)$.

d. Find $\operatorname{rank}(A)$.

9. Let $V = \operatorname{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$, where $\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -2 \\ -3 \\ -1 \\ 4 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 2 \end{pmatrix}$, and $\vec{v}_4 = \begin{pmatrix} 4 \\ 5 \\ 3 \\ -6 \end{pmatrix}$.

a. Find a basis for V.

Answer: *The column space of the matrix*

$$A = \begin{pmatrix} 2 & -2 & 0 & 4 \\ 1 & -3 & -1 & 5 \\ 3 & -1 & 1 & 3 \\ 0 & 4 & 2 & -6 \end{pmatrix}$$

is the row space of its transpose A^T. The rref of A^T is

$$\text{rref}\begin{pmatrix} 2 & 1 & 3 & 0 \\ -2 & -3 & -1 & 4 \\ 0 & -1 & 1 & 2 \\ 4 & 5 & 3 & -6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

A basis is of V is therefore given by

$$\{\begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \end{pmatrix}\}.$$

b. Is the vector $\left(\begin{smallmatrix} 5 \\ 0 \\ 1 \\ 2 \end{smallmatrix}\right)$ in V?

Answer: *No. If*

$$x\begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 1 \\ 2 \end{pmatrix},$$

then $x = 5$ and $y = 0$, which contradicts $2x - y = 1$.

10. Let V be the subspace of $\mathbb{R}^4$ which consists of all multiples of the vector $\left(\begin{smallmatrix} 1 \\ 2 \\ -1 \\ 3 \end{smallmatrix}\right)$. Find a basis for $V^\perp$.

Answer: *The space $V^\perp$ is the colloection of vectors (x_1, x_2, x_3, x_4) for which $x_1 + 2x_2 - x_3 + 3x_4 = 0$, i.e.,*

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2x_2 + x_3 - 3x_4 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

A basis is of $V^\perp$ is therefore given by

$$\{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \end{pmatrix}\}.$$

11. Let V be the subspace of $\mathbb{R}^3$ with basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$. Find the $\mathcal{B}$-coordinate vector $[\vec{v}]_{\mathcal{B}}$ for $\vec{v} = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$.

Answer: *This problem can be easily converted into a system of* 3 *equations in variables* x, y*, having augmented matrix* $\begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 5 \\ -1 & 0 & 2 \end{pmatrix}$*. The rref of this matrix is* $\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$*, which leads to* $x = -2$ *and* $y = 5$*. Indeed, one can check that*

$$-2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + 5 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}.$$

Therefore, $\mathcal{B}$*-coordinate vector is* $[\vec{v}]_{\mathcal{B}} = (-2, 5)$*.*

12. Use the Gram-Schmidt process to find an orthonormal basis for the subspace V of $\mathbb{R}^3$ spanned by the vectors $\vec{v}_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$.

Answer: *The Gram-Schmidt process is an iterative method to compute vectors* $\vec{b}_1 = \vec{v}_1$ *and* $\vec{b}_2 = \vec{v}_2 - a_1\vec{v}_1$*, such that* $\vec{b}_1 \cdot \vec{b}_2 = 0$*. As a final step, we normalize them so that they have length* 1*. The condition* $\vec{b}_1 \cdot \vec{b}_2 = 0$ *implies that* $a_1 = \vec{v}_1 \cdot \vec{v}_2 / ||\vec{v}_1||^2 = 9/9 = 1$*, so we take* $\vec{b}_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$*,* $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}$ *. Normalizing them to have length* 1 *gives*

$$\left\{ \begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{18} \\ 1/\sqrt{18} \\ -4/\sqrt{18} \end{pmatrix} \right\}.$$

This is the orthonormal basis of V *which is asked for.*

13. Let V be the subspace of $\mathbb{R}^3$ spanned by $\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

a. Find the matrix of the orthogonal projection of $\mathbb{R}^3$ onto V.

Answer: *Call this matrix* P*. Since the vectors* $v_1 = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$ *and* $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ *are orthogonal, we can use the orthogonal projection formula*

$$pr_V(v) = \frac{v_1 \cdot v}{||v_1||^2} v_1 + \frac{v_2 \cdot v}{||v_2||^2} v_2,$$

plugging in for v the standard basis vectors to give the matrix form $P = [pr_V(e_1), pr_V(e_2), pr_V(e_3)]$. We compute

$$pr_V(e_1) = \frac{v_1 \cdot e_1}{||v_1||^2} v_1 + \frac{v_2 \cdot e_1}{||v_2||^2} v_2 = \begin{pmatrix} 9/25 \\ 0 \\ 12/25 \end{pmatrix},$$

$$pr_V(e_2) = \frac{v_1 \cdot e_2}{||v_1||^2} v_1 + \frac{v_2 \cdot e_2}{||v_2||^2} v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$$pr_V(e_3) = \frac{v_1 \cdot e_3}{||v_1||^2} v_1 + \frac{v_2 \cdot e_3}{||v_2||^2} v_2 = \begin{pmatrix} 12/25 \\ 0 \\ 16/25 \end{pmatrix}.$$

Therefore,

$$P = \begin{pmatrix} 9/25 & 0 & 12/25 \\ 0 & 1 & 0 \\ 12/25 & 0 & 16/25 \end{pmatrix}.$$

b. Find the orthogonal projection of the vector $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ onto V.

Answer: *It is easy to see that if $b = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ then the projection formula gives the answer immediately:*

$$pr_V(b) = \frac{v_1 \cdot b}{||v_1||^2} v_1 + \frac{v_2 \cdot b}{||v_2||^2} v_2 = \begin{pmatrix} 9/25 \\ 2 \\ 12/25 \end{pmatrix}.$$

Alternatively, you can also multiply out Pb to get the answer.

14. Find the linear function $a + bt$ which best fits the data points $(0, 4)$, $(1, 5)$ and $(2, 0)$ using least squares.

Answer: *We first formulate this as an orthogonal projection problem. We would very much like it to be true that these points all lie on the same line. In this case, we would have that all these points (x, y) satisfy $y = a + bx$. In this case, we would have the simultaneous equations*

$$4 = a + 0 \cdot b, \quad 5 = a + 1 \cdot b, \quad 0 = a + 2 \cdot b,$$

which are equivalent to the matrix equation $Ax = r$, where

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad x = \begin{pmatrix} a \\ b \end{pmatrix}, \quad r = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}.$$

The first two coordinates give us conditions implying $a = 4$, $a + b = 5$, therefore $b = 1$. Sadly, this does **not** *satisfy the last condition $a + 2b = 0$. Therefore, this system cannot be solved. However, least squares gives you the* **closest** *possible solution. The idea is to project the vector r onto the space generated by the column vectors of A (thus guaranteeing us a closest possible solution). To do this, we use the "least squares projection formula" $x = (A^T \cdot A)^{-1} \cdot Ar$,*

$$(A^T \cdot A)^{-1} = \begin{pmatrix} \frac{5}{6} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

so

$$(A^T \cdot A)^{-1} \cdot A^T r = \begin{pmatrix} \frac{5}{6} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}.$$

This says $a = 5$ and $b = -2$, so the line $y = -2x + 5$ best fits the data points $(0, 4)$, $(1, 5)$ and $(2, 0)$.

Alternative solution: *This uses only calculus. You want to minimize the distance-squared function*

$$d(a, b) = (y_1 - bx_1 - a)^2 + (y_2 - bx_2 - a)^2 + (y_3 - bx_3 - a)^2 = (4 - a)^2 + (5 - b - a)^2 + (0 - 2b - a)^2,$$

where (x_i, y_i) are the points $(0, 4)$, $(1, 5)$ and $(2, 0)$. At a minimum, the gradient must be $(0, 0)$, so we compute the partial derivatives and set them equal to 0,

$$\partial d/\partial a = -18 + 6a + 6b = 0, \qquad \partial d/\partial b = -15 + 6a + 10b = 0.$$

Solving these two equations gives $a = 5$ and $b = -2$ as before.

15. Suppose that $T : \mathbb{R}^5 \mapsto \mathbb{R}^8$ is a linear transformation and $\dim(\ker(T)) = 2$. What are the possible values of $\dim(\mathrm{Image}(T))$?

Answer: *By the "fundamental theorem of linear algebra," we have dim(ker(T)+dim(Image(T))=number of columns = 5, so dim(Image(T))=5-2=3.*

In Problems 16-19, do all calculations by hand and show all steps. You may check your work with a calculator.

16. Let A be the matrix

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}.$$

a. Find the characteristic polynomial $f_A(\lambda)$ of A.

Answer: *The characteristic polynomial is* $f_A(z) = z^2 - 7z + 10$.

b. Find the eigenvalues of A. Find the algebraic multiplicity of each eigenvalue.

Answer: *The characteristic polynomial factors as* $(z-2)(z-5)$, *so* A *has eigenvalues* $\lambda_1 = 2$, $\lambda_2 = 5$.

c. For each eigenvalue λ, find a basis for the eigenspace E_λ.

Answer: *The eigenspaces are all* 1*-dimensional. The eigenvalue* $\lambda_1 = 2$ *has eigenvector* $v_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. *The eigenvalue* $\lambda_2 = 5$ *has eigenvector* $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

d. Is A diagonalizable? If so, find an eigenbasis for A and a diagonal matrix D which is similar to A. If not, give your reasons and evidence.

Answer: *Yes. An eigenbasis is* $\mathcal{B} = \{v_1, v_2\}$, *where* v_i *is as above.*

17. Repeat Problem 16 for

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Answer: *In this case,* $\lambda_1 = -2$, $\lambda_2 = \lambda_3 = 2$, *and* E_{-2} *is* 1*-dimensional, whereas* E_2 *is* 2*-dimensional. A basis of* E_{-2} *is* $\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\}$, *whereas a basis of* E_2 *is* $\{\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\}$.

18. Repeat Problem 16 for

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Answer: *In this case,* $\lambda_1 = \lambda_2 = \lambda_3 = 1$, *and* $E_1 = \ker(A - I)$ *is 2-dimensional. A basis of* E_2 *is* $\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \}$.

19. a. Find the 4-volume of the parallelepiped in $\mathbb{R}^4$ formed from the vectors

$$\begin{bmatrix} 2 \\ 2 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 1 \\ 7 \end{bmatrix}.$$

Answer: $|\det \begin{pmatrix} 2 & 0 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 4 & -4 & 1 & 1 \\ 2 & 0 & 5 & 7 \end{pmatrix}| = 16.$

b. Use Cramer's Rule to solve the linear algebraic system below. No credit for other methods.

$$\begin{aligned} 2x_1 + 3x_3 + 4x_4 &= 1 \\ 2x_1 + x_3 + 2x_4 &= 0 \\ 4x_1 - 4x_2 + x_3 + x_4 &= 0 \\ 2x_1 + 5x_3 + 7x_4 &= 0. \end{aligned}$$

Answer: *If*

$$A_1 = \begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & -4 & 1 & 1 \\ 0 & 0 & 5 & 7 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 1 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 4 & 0 & 1 & 1 \\ 2 & 0 & 5 & 7 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 2 & 0 & 1 & 4 \\ 2 & 0 & 0 & 2 \\ 4 & -4 & 0 & 1 \\ 2 & 0 & 0 & 7 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 2 & 0 & 3 & 1 \\ 2 & 0 & 1 & 0 \\ 4 & -4 & 1 & 0 \\ 2 & 0 & 5 & 0 \end{pmatrix},$$

then

$$x_1 = \frac{\det(A_1)}{\det(A)} = 3/4, \; x_2 = \frac{\det(A_2)}{\det(A)} = 7/8,$$
$$x_3 = \frac{\det(A_3)}{\det(A)} = 5/2, \; x_4 = \frac{\det(A_4)}{\det(A)} = -2.$$

c. Let $T\vec{x} = A\vec{x}$ be the linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^4$ where A is the matrix whose columns are the 4 vectors shown above in part **(a).** If we apply T to a cube C whose 4-volume is 6, what will be the volume of the image?

Answer: $\mathrm{vol}(T(C)) = |\det(T)| \cdot \mathrm{vol}(C) = 16 \cdot 6 = 96.$

20. True or False. If true, give a reason. If false, give a reason or a counterexample.

a. If A is a square matrix and $A^4 = I$, then A is invertible.

Answer: *True. The determinant of A must be non-zero, so it has to be invertible.*

b. Any set of vectors which spans $\mathbb{R}^n$ is a subset of some basis of $\mathbb{R}^n$.

Answer: *False. The spanning vectors could be linearly dependent, in which case they cannot all belong to a basis.*

c. Every square matrix has at least one real eigenvalue.

Answer: *False. For example,* $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *has eigenvalues* $\pm i$.

d. If V is the set of all vectors $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ in $\mathbb{R}^2$ with $|x| = |y|$, then V is a subspace of $\mathbb{R}^2$.

Answer: *False.* $\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) + \left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right) = \left(\begin{smallmatrix}2\\0\end{smallmatrix}\right)$ *is not in* V*. Therefore,* V *is not closed under addition.*

e. If $T\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) = \left(\begin{smallmatrix}x+y+1\\x+y-1\end{smallmatrix}\right)$, then T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Answer: *False.* $T\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right) \neq \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$*. Therefore,* $T(0 \cdot \vec{x}) = 0 \cdot T(\vec{x}) = \vec{0}$ *fails to hold.*

END OF TEST