

**Applications of Fourier Transforms and Convolutions in Optics: Limited to the  
Application of the Diffraction of Light**

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**Goals:** To examine the applications of Fourier transforms and convolutions in Fourier optics with specific attention devoted to the Fourier lens. The fundamental properties of Fourier transformations coupled with convolutions are an integral part of Fourier optics. Several properties of Fourier transforms such as are reviewed and directly related to actual optical demonstrations in the laboratory.

Nevertheless, before delving into the complex realm of Fourier optics, a certain amount of mathematical introduction and explanation is required. The theorems, definitions, and proofs are presented to introduce the fundamentals necessary for expansion from theoretical to practical. But who was Fourier?



Joseph Fourier had been quite gifted in mathematics early in his teenage years. However, he had decided embark upon a career in the priesthood in France. He was very conflicted between mathematics and religion and would ultimately deny his religious vows. Fourier became entangled in the French revolution and found himself placed in prison awaiting the guillotine. But, he was eventually freed. Fourier then began teaching at the Collège de France and worked side by side with Lagrange and Laplace. After teaching, Fourier joined Napoleon's Army and was in charge of new innovations in Egypt. While in Egypt, Fourier derived the heat equation. After his military service, which was closely dictated by Napoleon until his defeat, he returned to France and continued his research. Fourier was not without controversy. In fact, Jean-Baptiste Biot had attempted to claim the discovery of Fourier's heat theory and Poisson disputed his methods.<sup>1</sup>

<http://www-history.mcs.st-andrews.ac.uk/history/Posters2/Fourier.html>

What is Fourier analysis? Fourier analysis allows a signal or function to be separated or decomposed into components made up of simpler inputs for linear systems. The individual inputs have a response associated with them and that is the sum of the responses to the individual points.<sup>2</sup> The process described above is referred to as convolution or superposition in a linear system.

Still confused as to what convolution really is?

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<sup>1</sup> [MacTutor History of Mathematics](http://www-history.mcs.st-andrews.ac.uk/Biographies/Fourier.html), Jan. 1997, University of St. Andrews, 24 Jan. 2007 < <http://www-history.mcs.st-andrews.ac.uk/Biographies/Fourier.html> >.

<sup>2</sup> Joseph W. Goodman, [Introduction to Fourier Optics](#) (New York: McGraw-Hill, 1968) 7.

Steward explains convolution as, “the distribution of one function in accordance with the law specified by another function.”<sup>3</sup> We might say the smearing of one function with another. Essentially, each ordinate of a function is multiplied by another function and summed.<sup>4</sup> Nevertheless, the functions are assumed linear and invariant or “stationary” because transformations do not change the function, thus preserving length.<sup>5</sup> Convolutions are overlaps that are the result of spreading or smearing of one function using the rule or operation of another.

**Definition of convolution and theorem<sup>6</sup>**

If  $f$  and  $g$  are two functions, with  $\|f\|_1$  and  $\|g\|_1$  finite, the **convolution** of  $f$  and  $g$  is denoted by  $f * g$  and is defined by:

$$f * g(x) = \int_{-\infty}^{\infty} f(s) g(x - s) ds$$

But, convolutions are tremendously powerful because of the properties they possess.

**Such as :**

- 1) Commutivity-  $f * g = g * f$
- 2) Associativity-  $f * (g * h) = (f * g) * h$
- 3) Distribution of Addition-  $f * (g + h) = (f * g) + (f * h)$ <sup>7</sup>

We begin with an introduction of convolution over a finite interval using the Fourier series, which is later expanded to an infinite interval using the Fourier transform.

The function  $f * P$  is the **convolution over  $[-L, L]$  of  $f$  and  $P$ .**

Defined by:

$$f * P(x) = \frac{1}{2L} \int_{-L}^L f(s) P(x - s) ds$$
<sup>8</sup>

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<sup>3</sup> E.G. Steward, Fourier Optics: an Introduction, 2<sup>nd</sup> Ed. (New York: Halstead, 1987) 85.

<sup>4</sup> Steward 85-86.

<sup>5</sup> Steward 102.

<sup>6</sup> Walker 173.

<sup>7</sup> Steward 85.

<sup>8</sup> James Walker, Fast Fourier Transforms, 2<sup>nd</sup> Ed. (Boca Raton: CRC, 1987) 117.

**PROOF**<sup>9</sup>

Given  $f * P$  exists, compute the  $n^{\text{th}}$  Fourier coefficient,  $p_n$

$$\begin{aligned} p_n &= \frac{1}{2L} \int_{-L}^L f * P(x) e^{-i\pi nx/L} dx \\ &= \frac{1}{2L} \int_{-L}^L \left[ \frac{1}{2L} \int_{-L}^L f(s) P(x-s) ds \right] e^{-i\pi nx/L} dx \end{aligned}$$

However,  $e^{-i\pi nx/L}$  is not effected by  $\int_{-L}^L f(s) P(x-s) ds$ , and we can reverse the order of integration because the integrals are assumed to be convergent, and the integrations are over finite intervals. The limits are constants, that is, no limit depends on the other integration variable so the order may be interchanged. Thus becoming:

$$p_n = \frac{1}{2L} \int_{-L}^L f(s) \left[ \frac{1}{2L} \int_{-L}^L P(x-s) e^{-i\pi nx/L} dx \right] ds$$

Now substitute  $(x-s) + s$  into  $e^{-i\pi nx/L}$  and factor out  $e^{-i\pi ns/L}$

$$p_n = \frac{1}{2L} \int_{-L}^L f(s) e^{-i\pi ns/L} \left[ \frac{1}{2L} \int_{-L}^L P(x-s) e^{-i\pi n(x-s)/L} dx \right] ds$$

Now, complete a change of variables and let  $(x-s) = v$ :

$$p_n = \frac{1}{2L} \int_{-L}^L f(s) e^{-i\pi ns/L} \left[ \frac{1}{2L} \int_{-L-s}^{L-s} P(v) e^{-i\pi nv/L} dv \right] ds$$

This becomes:

$$p_n = \frac{1}{2L} \int_{-L}^L f(s) e^{-i\pi ns/L} \left[ \frac{1}{2L} \int_{-L}^L P(v) e^{-i\pi nv/L} dv \right] ds$$

Which is true because  $P(v) e^{-i\pi nv/L}$  has a period of  $2L$  so  $\int_{-L-s}^{L-s}$  becomes  $\int_{-L}^L$

However, we defined  $\frac{1}{2L} \int_{-L}^L P(v) e^{-i\pi nv/L} dv$  above as  $F_n$ :

$$F_n = \frac{1}{2L} \int_{-L}^L P(x) e^{-i\pi nx/L} dx \text{ and } p_n = \frac{1}{2L} \int_{-L}^L f(s) e^{-i\pi ns/L} F_n ds$$

But, recall  $c_n = \frac{1}{2L} \int_{-L}^L f(x) e^{-i\pi nx/L} dx$

Therefore,  $p_n = c_n F_n$

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<sup>9</sup> Walker 117-118.

**PROOF**<sup>10</sup> (The Convolution Theorem and its Applications (as seen on page 3))

$$\begin{aligned}\mathfrak{F}(f(x) * g(x)) &= \mathfrak{F}\left(\int_{-\infty}^{\infty} f(x)g(u-x) dx\right) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)g(u-x) dx e^{2\pi i s u} du\end{aligned}$$

Let  $w = u - x$

$$\begin{aligned}\mathfrak{F}(f(x) * g(x)) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)g(w) e^{2\pi i s(x+w)} dx dw \\ &= \mathfrak{F}(f)\mathfrak{F}(g) = \hat{f}(u) \hat{g}(u)\end{aligned}$$

However, the link between the Fourier transform and convolution becomes evident with the introduction of the convolution theorem below.

**Convolution Theorem**<sup>11</sup>: The Fourier Transform of the convolution of two functions  $f$  and  $g$  is the multiplication of the Fourier transform of  $f$  with the Fourier transform of  $g$ .

$$f * g \text{ is } (\hat{f})(\hat{g})$$

The convolution theorem is a very powerful statement because of the relationship it forms between Fourier transforms and convolution. The theorem states that the Fourier transform of a convolution is the product of the Fourier transforms of each of the individual functions. The convolution theorem allows a convolution of two functions to be taken by multiplying the Fourier transform of the respective functions, thus simplifying an immense amount of calculation. However, the result must have a Fourier inverse.

With a slight modification, the statement relates the inverse Fourier transform of two functions to the convolution of two functions. The theorem is just as powerful as the previous version because it reveals that whenever an inverse Fourier transform is done on two functions, there is a corresponding convolution of two functions. The theorem unlocks a very important and complex idea that's within our context.

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<sup>10</sup> Hecht 491.

<sup>11</sup> Walker 178.

**Convolution Theorem<sup>12</sup> (Inverse) :** The inverse Fourier transform of  $\hat{f} \hat{g}$  is the convolution of  $f * g$

Such a remarkable theorem requires a proof because of its possibilities and applications in mathematics, most specifically optics.

**PROOF<sup>13</sup>**

Using the inverse Fourier Transform:

$$\begin{aligned} \mathfrak{S}^{-1}(\hat{f} \hat{g})(x) &= \int_{-\infty}^{\infty} \hat{f}(u) \hat{g}(u) e^{-2i\pi ux} du \\ &= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(s) e^{-2i\pi us} ds \right] \hat{g}(u) e^{-2i\pi us} du \end{aligned}$$

Now interchange the integrals and add the exponents:

$$\mathfrak{S}^{-1}(\hat{f} \hat{g})(x) = \int_{-\infty}^{\infty} f(s) \left[ \int_{-\infty}^{\infty} \hat{g}(u) e^{-2i\pi u(x-s)} du \right] ds$$

This is the integral of  $\hat{g}(u)$ , with the variable  $(x-s)$  so we get  $g(x-s)$  for that integral.

Thus,  $\mathfrak{S}^{-1}(\hat{f} \hat{g})(x) = f * g$

Change the sum exponentials to a product of exponentials

$$\begin{aligned} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) e^{2\pi isx} g(w) e^{2\pi isw} dx dw \\ &= \int_{-\infty}^{\infty} f(x) e^{2\pi isx} dx \int_{-\infty}^{\infty} g(w) e^{2\pi isw} dw \end{aligned}$$

$w$  is a dummy variable, so replace  $w$  with  $x$

$$\mathfrak{S}(f(x) * g(x)) = \int_{-\infty}^{\infty} f(x) e^{2\pi isx} dx \int_{-\infty}^{\infty} g(x) e^{2\pi isx} dw$$

Thus,  $\mathfrak{S}(f(x) * g(x)) = \mathfrak{S}(f(x)) \mathfrak{S}(g(x)) = (\hat{f})(\hat{g})$

As shown, convolutions are overlaps that are the result of spreading or smearing of one function by another. Convolution facilitates representing a regular complex pattern to as a basic pattern repeated in accordance with an array function. Nevertheless, all the power does not rest in convolution alone, but jointly between convolution and Fourier transforms because in our context, they are tightly interlaced.

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<sup>12</sup> Walker 178.

<sup>13</sup> Walker 177

What is a Fourier transform? A Fourier transform represents a function as a sum of characteristic frequencies or pure frequencies. But, the caveat is that Fourier transforms are used with Lebesgue square integrable functions. Physicists normally refer to the Fourier transform as the Fourier spectrum and tend to look at its two-dimensional version where light is illuminated on a plane screen perpendicular to the direction of propagation.<sup>14</sup> A standard convention is to run the coordinate  $z$  in the direction of the propagation and to use  $x$  and  $y$  in the plane of the screen.

**Definition of Fourier Transform**<sup>15</sup>

Let  $\|f\|_1$  be defined as  $[\int_{-\infty}^{\infty} |f(x)| dx]^{\frac{1}{2}}$

Given a function for which  $\|f\|_1$  is finite, the **Fourier transform** of  $f$  is denoted by  $\hat{f}$  and is defined as a function of  $u$ :

$$\hat{f}(u) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi ux} dx$$

Gaussian functions ( $\sqrt{\frac{a}{\pi}} e^{-ax^2}$ ) are of particular interests because the Fourier transform of a Gaussian function is another Gaussian function. The Gaussian and its Fourier transform have the same relative shape and is a good example for studying the relative widths of a function and its transform. This result is of great interest to physicists as a ‘good’ or ‘nice’ laser beam has a transverse profile that is Gaussian. The Gaussian profile diffracts into a Gaussian profile and essentially projects itself, thus maintaining its relative transverse profile. Note the derivation below to see for yourselves.

**Example**<sup>16</sup>

let  $f(x) = \sqrt{\frac{a}{\pi}} e^{-ax^2}$ ,  $\mathcal{F}\{f(x)\} = ?$

$F(k) = \int_{-\infty}^{\infty} (\sqrt{\frac{a}{\pi}} e^{-ax^2}) e^{ikx} dx$ , now complete the square:

$$-ax^2 + ikx \text{ becomes } -(x\sqrt{a} - ikx)^2 - \frac{k^2}{4}$$

Now let  $(x\sqrt{a} - ikx) = \beta$ , so

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<sup>14</sup> Goodman 5, 86.

<sup>15</sup> Walker 149-150.

<sup>16</sup> Eugene Hecht, Optics, 2<sup>nd</sup> Ed. (Reading MA, Addison-Wesley) 474.

Square the integral so we can place it into polar form:

$$\frac{\sqrt{\frac{a}{\pi}}}{\sqrt{a}} e^{-\frac{k^2}{4a}} \left( \int_{-\infty}^{\infty} e^{-\beta^2} d\beta \right) \left( \int_{-\infty}^{\infty} e^{-\beta^2} d\beta \right) \text{ let the second } \beta = \alpha$$

$$F(k) = \int_{-\infty}^{\infty} \sqrt{\frac{a}{\pi}} e^{-\beta^2 - \frac{k^2}{4a}} d\beta \Rightarrow F(k) = \frac{\sqrt{\frac{a}{\pi}}}{\sqrt{a}} e^{-\frac{k^2}{4a}} \int_{-\infty}^{\infty} e^{-\beta^2} d\beta$$

$$= \frac{\sqrt{\frac{a}{\pi}}}{\sqrt{a}} e^{-\frac{k^2}{4a}} \left( \int_{-\infty}^{\infty} e^{-\beta^2} d\beta \right) \left( \int_{-\infty}^{\infty} e^{-\alpha^2} d\alpha \right)$$

$$= \frac{\sqrt{\frac{a}{\pi}}}{\sqrt{a}} e^{-\frac{k^2}{4a}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\beta^2 - \alpha^2} d\beta d\alpha$$

$$= \beta^2 + \alpha^2 = r^2, d\beta d\alpha = r dr d\theta$$

$$= \frac{\sqrt{\frac{a}{\pi}}}{\sqrt{a}} e^{-\frac{k^2}{4a}} \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta$$

$$= \frac{\sqrt{\frac{a}{\pi}}}{\sqrt{a}} e^{-\frac{k^2}{4a}} \left( \int_0^{2\pi} d\theta \right) \left( \int_0^{\infty} e^{-r^2} dr \right)$$

let  $u = r^2$ ,  $du = 2r dr$ , so  $r dr = \frac{1}{2} du = dr$

$$\left( \int_0^{2\pi} d\theta \right) \left( \frac{1}{2} \int_0^{\infty} e^{-u^2} du \right) = \pi, \text{ but remember we squared the integral } \Rightarrow \sqrt{\pi}$$

$$= \left( \frac{\sqrt{\frac{a}{\pi}}}{\sqrt{a}} e^{-\frac{k^2}{4a}} \right) (\sqrt{\pi})$$

$$\Rightarrow \frac{\sqrt{\pi} \sqrt{\frac{a}{\pi}}}{\sqrt{a}} e^{-\frac{k^2}{4a}} = e^{-\frac{k^2}{4a}}$$

**Note** : the integration of  $\sqrt{\frac{a}{\pi}} e^{-\beta^2}$  is possible along a contour parallel to the real axis because it can be shifted to a contour along the real axis.

With the Fourier transform defined, the inverse seems to follow naturally. The inverse Fourier transform is actually the Fourier transform with positive exponent.<sup>17</sup>

### Definition of Inverse Fourier Transform<sup>18</sup>

If  $f$  is continuous and  $\|f\|_1$  and  $\|\hat{f}\|_1$  are both finite, then

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(u) e^{i2\pi ux} dx \text{ for all } x \in \mathbb{R}$$

<sup>17</sup> Walker 160.

<sup>18</sup> Walker 160.

As previously mentioned, convolution and Fourier transforms are interrelated in our context. However, with a slight modification to the definition of convolution, the relation is easily and noticeably applicable. Be aware, that convolution is also referred to as “the folding product” and “composition product” by physicists.<sup>19</sup>

Fourier transforms have four very important properties which allow for simplification and useful manipulation in both physics and mathematics. These properties are linearity, scaling, shifting, and modulation, which are defined on the following page.

**Linearity Theorem**<sup>20</sup> – For  $\forall a, b$  where  $a$  and  $b$  are constants, the Fourier transform satisfies  $af + bg \xrightarrow{\mathcal{S}} a\hat{f} + b\hat{g}$

**Scaling Theorem** – For each positive constant  $p$ ,

$$f\left(\frac{x}{p}\right) \xrightarrow{\mathcal{S}} p\hat{f}(pu) \text{ and } f(px) \xrightarrow{\mathcal{S}} \frac{1}{p}\hat{f}\left(\frac{u}{p}\right)$$

**Modulation Theorem** – For each real constant  $c$ ,

$$f(x)e^{-i2\pi cx} \xrightarrow{\mathcal{S}} \hat{f}(u - c)$$

**Shift Theorem** – For each real constant  $c$ ,

$$f(x - c) \xrightarrow{\mathcal{S}} \hat{f}(u)e^{-i2\pi cu}$$

**PROOF**<sup>21</sup> (Scaling Theorem)

$$\text{let } s = \frac{x}{p} \text{ so } f\left(\frac{x}{p}\right) \xrightarrow{\mathcal{S}} \int_{-\infty}^{\infty} f\left(\frac{x}{p}\right) e^{-i2\pi ux} dx = \int_{-\infty}^{\infty} f(s) e^{-i2\pi u(ps)} d(ps)$$

$$= p \int_{-\infty}^{\infty} f(s) e^{-i2\pi (pu)s} ds = p\hat{f}(pu)$$

The second part of the proof follows with minor manipulation.

**PROOF**<sup>22</sup> (Modulation Theorem)

note  $e^{i2\pi cx} e^{-i2\pi ux} = e^{-i2\pi(u-c)x}$  so

$$f(x)e^{i2\pi cx} \xrightarrow{\mathcal{S}} \int_{-\infty}^{\infty} f(x) e^{-i2\pi(u-c)x} dx = \hat{f}(u - c)$$

<sup>19</sup> Steward 82.

<sup>20</sup> Walker 155.

<sup>21</sup> Walker 156.

<sup>22</sup> Walker 156.

**PROOF<sup>23</sup> (Shift Theorem)**

$$\begin{aligned} \text{let } s = x - c \text{ so } f(x - c) &\xrightarrow{\mathfrak{F}} \int_{-\infty}^{\infty} f(x - c) e^{-i2\pi ux} dx = \int_{-\infty}^{\infty} f(s) e^{-i2\pi u(s+c)} ds \\ &= \int_{-\infty}^{\infty} f(s) e^{-i2\pi us} ds e^{-i2\pi cu} = \hat{f}(u) e^{-i2\pi cu} \end{aligned}$$

The modulation theorem is very advantageous in optics because it shows that multiplication by a linear phase creates a translation and the shift theorem shows that the inverse is true. When a translation is multiplied by a linear phase as in the case of the Gaussian, the result is another Gaussian.<sup>24</sup>

**Example of a Gaussian [Shift theorem]:**

Start with:

$$\int_{-\infty}^{\infty} e^{i\lambda x} f(x + a) dx \xrightarrow{\mathfrak{F}} e^{ika} \hat{f}(k - \lambda)$$

Thus, multiplying by a linear phase creates a translation

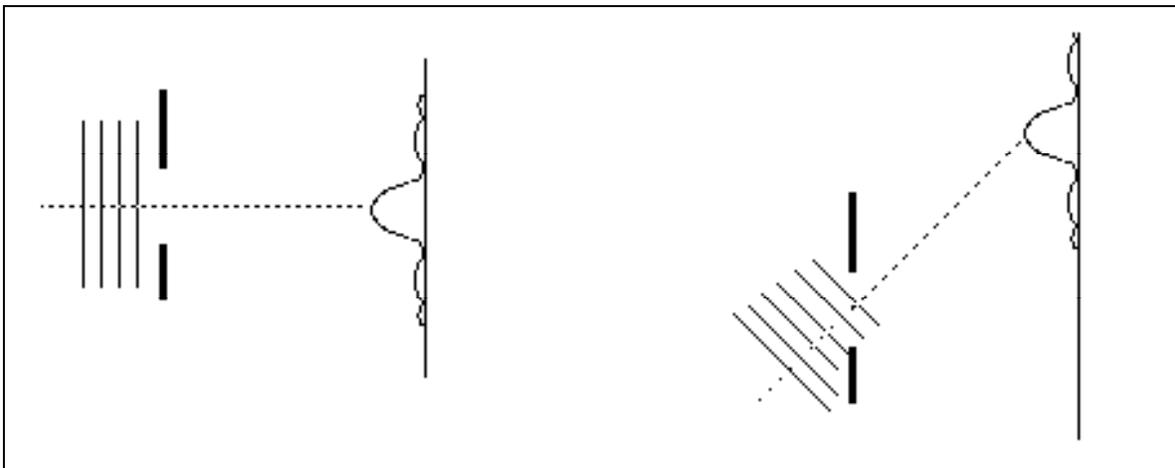
Then,

$$\int_{-\infty}^{\infty} e^{\frac{-x^2}{2\Delta}} e^{ikx} dx \xrightarrow{\mathfrak{F}} e^{\frac{-k^2\Delta^2}{2}}$$

Now we can use the above to conclude

$$\int_{-\infty}^{\infty} e^{\frac{-(x^2-a)^2}{2\Delta^2}} e^{ikx} dx \xrightarrow{\mathfrak{F}} e^{ika} \hat{f}(k - k_0) = e^{\frac{-(k-k_0)^2\Delta^2}{2}}$$

Seemingly difficult transformations become manageable.



**Figure 1<sup>25</sup>**

<sup>23</sup> Walker 156.

<sup>24</sup> Hecht 474.

<sup>25</sup> Diagram obtained from Prof. Tankersley

Figure 1 shows a plane wave entering through an aperture. The diffraction pattern of the plane wave through the slit is the square of Fourier transform, which explains why the pattern has only positive components. If the light is shifted at the aperture, the result is a modulation of the square of the Fourier transform. Nevertheless, the entering light maintains the same amplitude (square of the amplitude), visibly displaying Parseval's equality and properties of Fourier transforms.

Fourier transformations themselves are a necessary tool for physicists and mathematicians because of their properties, but coupling with "convolution" allows for remarkable, tangible results in physics and mathematics as seen below. The diagram depicts the top-hat function convolved with another top-hat function creating the triangle function which is precisely how a spectrometer works.

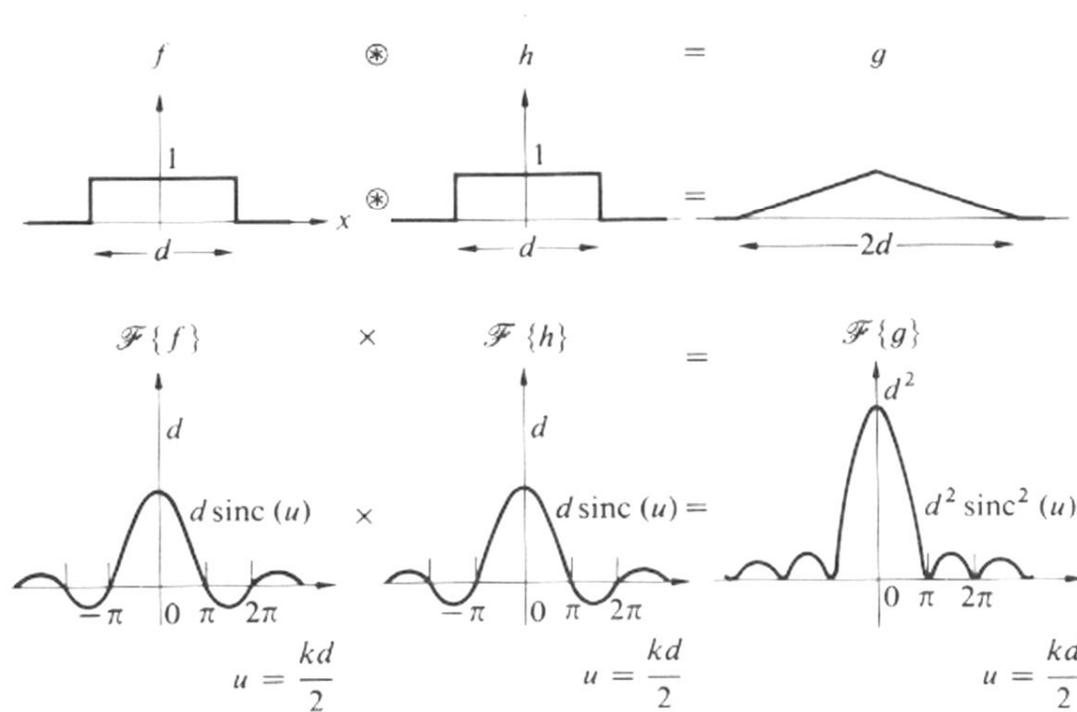


Figure 2<sup>26</sup>

<sup>26</sup> Hecht 492.

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