Fourier Transforms and Convolutions

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Who Was Fourier??

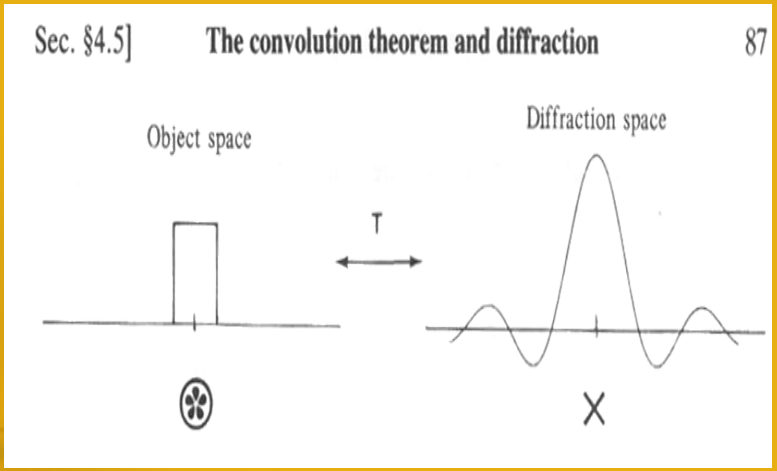
- Noticeably gifted by age 14
- Priest or Mathematician?
- Math it is
- Taught at the Collège de France
- Joined Napoleon's Army
- OIC of discoveries in Egypt
- Poisson and Biot

What is Fourier Analysis?

- Fourier analysis allows a system to be separated or decomposed into components made up of simpler inputs.
- For example, a function f(t) is a function in time but via a Fourier transform becomes f(w) , where omega is a frequency

Convolution:

- Steward describes convolution as "the distribution of one function in accordance with the law specified by another function (85)."



Convolution cont.

- Overlaps that are a result of spreading and smearing of a function

Definition of Convolution

Definition of convolution and theorem

If f and g are two functions, with $\|f\|_1$ *and* $\|g\|_1$ *finite, the* **convolution** *of f and g is denoted by* $f*g$ *and is defined by.*

$$f*g(x) = \int_{-\infty}^{\infty} f(s)\, g(x-s)\, ds \quad pg\ 173$$

PROOF *(The Convolution Theorem and its Applications)*

$$*(f(x) = g(x)) \quad (\int_{-\infty}^{\infty} f(x)\, g(u) \quad x)\, dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\, g(u-x)\, dx\, e^{2\pi i s u}\, du$$

Let $w = u - x$

$$*(f(x) = g(x)) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\, g(w)\, e^{2\pi i s(x+w)}\, dx\, dw$$

Convolution Theorem

Convolution Theorem . *The Fourier Transform of the convolution of two functions f and g is the multiplication of the Fourier transform of f with the Fourier transform of g*

$f * g$ *is* $\hat{f}\hat{g}$

Convolution Theorem Inverse : *The inverse Fourier transform of* $\hat{f}\hat{g}$ *is the convolution of* $f * g$

PROOF

Using the inverse Fourier Transform.

$$= \mathcal{F}^{-1}\left(\hat{f}\hat{g}\right)(x) \quad \int \hat{f}(u)\hat{g}(u)\, e^{-2i\pi ux}\, du$$

$$= \int \left(\int f(s)\, e^{-2i\pi us}\, ds\right)\hat{g}(u)\, e^{-2i\pi us}\, du$$

Now interchange the integrals and add the exponents. pg 177

$$= \mathcal{F}^{-1}\left(\hat{f}\hat{g}\right)(x) \quad \int f(s)\left(\int \hat{g}(u)\, e^{-2i\pi u(x-s)}\, du\right) ds$$

This is the integral of $\hat{g}(u)$ *with the variable* $(x-s)$ *so we get* $g(x-s)$ *for that integral*

Thus $= \mathcal{F}^{-1}\left(\hat{f}\hat{g}\right)(x) \quad f \quad g$

Change the sum exponentials to a product of exponentials

$$= \int \int f(x)\, e^{2\pi isx}\, g(w)\, e^{2\pi isw}\, dx\, dw$$

$$= \int f(x)\, e^{2\pi isx}\, dx \int g(w)\, e^{2\pi isw}\, dw$$

w is a dummy variable so replace w with x

$$* \quad f(x) = g(x) \quad \int f(x)\, e^{2\pi isx}\, dx \int g(x)\, e^{2\pi isx}\, dw$$

Thus $* \quad f(x) = g(x) \quad f(x) = g(x) \quad \hat{f}\hat{g}$

Parseval's Equality

- Energy Conservation Statement

Theorem. (Walker 102)

We define $\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$ *as the Fourier sine series for the function* g

where $\|g\| < \infty$ *(finite) if and only if* $\sum_{n=1}^{\infty} |a_n|^2$ *is convergent. Thus introducing*

Parseval's equality :

$$\frac{L}{2}\sum_{n=1}^{\infty} |a_n|^2 = \int_0^L |g(x)|^2 dx = \|g\|^2$$

In physics, the relation is normally written as.

Given. $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(x) e^{ikx} dx$ *and* $\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$ *(Prof. Tankersley)*

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{g}(x) e^{imx} dx \text{ and } \hat{g}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-imx} dx$$

Then **Parseval's equality** *becomes.*

$$\int_{-\infty}^{\infty} \overline{g(x)} f(x)\, dx = \int_{-\infty}^{\infty} \overline{\hat{g}(k)}\, \hat{f}(k)\, dk$$ *where* $\overline{g(x)}$ *and* $\overline{f(x)}$ *is the comple conjugate.*