Applications of Fourier Transforms and Convolutions in Optics

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Thesis: To examine the applications of Fourier transforms and convolutions in Fourier optics with specific interest devoted to the Fourier lens. The immense properties of Fourier transformations coupled with convolutions are an integral part of Fourier optics. Several properties of Fourier transforms such as are reviewed and directly related to actual optical demonstrations in the laboratory.

Nevertheless, before delving into the complex realm of Fourier optics, there is a certain amount of mathematical introduction and explanation required. The theorems, definitions, and proofs provided introduce the fundamentals necessary for expansion from theoretical to practical. But who was Fourier?

Joseph Fourier had been quite gifted in mathematics early in his teenage years. However, he had decided embark upon the journey of priesthood in France. However, he was very conflicted between mathematics and religion and would ultimately deny his religious vows. Fourier became entangled in the French revolution and found himself placed in prison awaiting the guillotine. But, he was eventually freed. Fourier then began teaching at the Collège de France and worked side by side with Lagrange and Laplace. After teaching Fourier joined Napoleon's Army and was in charge of discoveries in Egypt. While in Egypt, Fourier was able to derive the heat equation. After his military service, which was closely dictated by Napoleon until his defeat, he returned to France and continued his research. Fourier was not without controversy. In fact, Jean-Baptiste Biot had attempted to claim the discovery of Fourier's heat theory and Poisson disputed his methods.

http://www-history.mcs.st-andrews.ac.uk/history/Posters2/Fourier.html

What is Fourier analysis? Fourier analysis allows a system to be separated or decomposed into components made up of simpler inputs. The individual inputs have a response associated with them and the combination of all the inputs is the total response (Goodman 7). The process described above is referred to as convolution or superposition in a linear system. Throughout the paper the term convolution should be synonymous with superposition.

Still confused as to what convolution really is?

Steward explains convolution as, "the distribution of one function in accordance with the law specified by another function (Steward 85)." Essentially, each ordinate of a function is multiplied by another function and summed (Steward 85). Nevertheless, the functions are assumed linear and invariant or "stationary" because transformations do not change the function (length preserving) (Steward 102). Convolutions are overlaps that are the result of spreading or smearing of one function using the rule or operation of another. But, convolutions are tremendously powerful because of the properties they possess.

Such as :

1) Commutivity- $f * g = g * f$

2) Associativity- $f * (g * h) = (f * g) * h$

3) Distribution of Addition- $f * (g + h) = (f * g) + (f * h)$ (Steward 85)

The function f ∗ P is the **convolution over** [-L,L] **of f and P**.

Defined by: (Walker 117)

$$f * P(x) = \frac{1}{2L}\int_{-L}^{L} f(s)\ P(x-s)\ ds$$

In optics, convolution in real space is interchangeable with multiplication in diffraction (Fourier space, frequency space, or diffraction space) (Steward 87).

PROOF :

Given $f * P$ exists, compute the nth Fourier coefficent, p_n

$$p_n = \tfrac{1}{2L}\int_{-L}^{L} f * P(x)\ e^{-i\pi nx/L}\ dx$$

$$= \tfrac{1}{2L}\int_{-L}^{L} [\tfrac{1}{2L}\int_{-L}^{L} f(s)\ P(x-s)\ ds]\ e^{-i\pi nx/L} dx$$

However, $e^{-i\pi nx/L}$ is not effected by $\int_{-L}^{L} f(s)\ P(x-s)\ ds$, and we can reverse the order of integration:

$$p_n = \tfrac{1}{2L}\int_{-L}^{L} f(s)[\tfrac{1}{2L}\int_{-L}^{L}\ P(x-s)\ e^{-i\pi nx/L} dx]\ ds$$

Now substitute (x-s) + s into $e^{-i\pi nx/L}$ and factor out $e^{-i\pi ns/L}$ pg. 118

$$p_n = \tfrac{1}{2L}\int_{-L}^{L} f(s)\ e^{-i\pi ns/L}[\tfrac{1}{2L}\int_{-L}^{L}\ P(x-s)\ e^{-i\pi n(x-s)/L} dx]\ ds$$

Now, complete a change of variables and let (x-s) = υ :

$$p_n = \tfrac{1}{2L}\int_{-L}^{L} f(s)\ e^{-i\pi ns/L}[\tfrac{1}{2L}\int_{-L-s}^{L-s}\ P(\upsilon)\ e^{-i\pi n\upsilon/L} d\upsilon]\ ds$$

This becomes:

$$p_n = \tfrac{1}{2L}\int_{-L}^{L} f(s)\ e^{-i\pi ns/L}[\tfrac{1}{2L}\int_{-L}^{L}\ P(\upsilon)\ e^{-i\pi n\upsilon/L} d\upsilon]\ ds$$

Which is true because P(υ) $e^{-i\pi n\upsilon/L}$ has a period of 2L so $\int_{-L-s}^{L-s}$ becomes $\int_{-L}^{L}$

However, we defined $\tfrac{1}{2L}\int_{-L}^{L}\ P(\upsilon)\ e^{-i\pi n\upsilon/L} d\upsilon$ above as F_n :

$F_n = \tfrac{1}{2L}\int_{-L}^{L} P(x)\ e^{-i\pi nx/L} dx$ and $p_n = \tfrac{1}{2L}\int_{-L}^{L}\ f(s)\ e^{-i\pi ns/L} F_n\ ds$

But, recall $c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-i\pi nx/L} dx$

Therefore, $p_n = c_n F_n$

As shown, the convolution is the sum of the multiplication of two functions f(s) and P(x-s) in our example before the proof. However, P(x-s) is shifted by -s. Convolution is an incredible tool used in both mathematics and physics because of the power it possesses. Convolution functions allow complex functions to be broken down and reassembled to give the total picture. Nevertheless, all the power does not rest in convolution alone, but jointly between convolution and Fourier transforms because in our context, they are tightly interlaced.

What is a Fourier transform? A Fourier transform takes a function that is dependent with respect to one variable (usually time and position in optics) and transforms that function into a function dependent upon another variable (usually frequency in optics). However, the caveat is that Fourier transforms are used with Lebesgue integrable functions. Physicists normally refer to the Fourier transform as the Fourier spectrum and tend to look at its two-dimensional version because light is generally examined in the X-Y plane (Goodman-5,86).

Definition of Fourier Transform

Let $\|f\|_1$ be defined as $[\int_{-\infty}^{\infty} |f(x)|^2 dx\,]^{\frac{1}{2}}$ pg. 149

Given a function for which $\|f\|_1$ is finite, the **Fourier transform** of f is denoted by $\hat{f}$ and is defined as a function of **u**:

$\hat{f}(u) = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi ux} dx$ pg. 150

The applications of Fourier transformations and convolutions are deeply rooted optics and acoustics, providing the foundation for Fraunhofer diffraction of scalar waves. It applies to acoustics and optics when the scalar approximation is valid. Therefore, there is some advantage in representing the Fourier transformation in terms of frequency (υ) and time (t). The frequency and time version of the Fourier transform is yet another powerful tool has immense capabilities in optics. In optics, light is expressed as a function of space for diffraction and with a Fourier transformation, the function of space becomes a function of frequency. The Fourier transform provides a way to describe the content of the plane waves. For example, in optics the Fourier transform represents a light signal as a sum of plane waves focused at a point on a screen. Even more interesting at the moment is that plane waves are normally explained with Gaussian functions ($\sqrt{\frac{a}{\pi}} e^{-ax^2}$) because the Fourier transform of a Gaussian function is another Gaussian function. Note the derivation below to see for yourselves.

Example : (Hecht 474)

let $f(x) = \sqrt{\frac{a}{\pi}} e^{-ax^2}$, $\Im\{f(x)\} = ?$

$F(k) = \int_{-\infty}^{\infty} (\sqrt{\frac{a}{\pi}} e^{-ax^2}) e^{ikx} dx$, now complete the square:

$-ax^2 + ikx$ becomes $-(x\sqrt{a} - ikx)^2 - \frac{k^2}{4}$

Now let $(x\sqrt{a} - ikx) = \beta$, so

$F(k) = \int_{-\infty}^{\infty} \sqrt{\frac{a}{\pi}} e^{-\beta^2 - \frac{k^2}{4a}} d\beta \Rightarrow F(k) = \frac{\sqrt{\frac{a}{\pi}}}{\sqrt{a}} e^{-\frac{k^2}{4a}} \int_{-\infty}^{\infty} e^{-\beta^2} d\beta$

Square the integral so we can place it into polar form:

$\frac{\sqrt{\frac{a}{\pi}}}{\sqrt{a}} e^{-\frac{k^2}{4a}} (\int_{-\infty}^{\infty} e^{-\beta^2} d\beta) (\int_{-\infty}^{\infty} e^{-\beta^2} d\beta)$ let the second $\beta = \alpha$

$= \frac{\sqrt{\frac{a}{\pi}}}{\sqrt{a}} e^{-\frac{k^2}{4a}} (\int_{-\infty}^{\infty} e^{-\beta^2} d\beta) (\int_{-\infty}^{\infty} e^{-\alpha^2} d\alpha)$

$= \frac{\sqrt{\frac{a}{\pi}}}{\sqrt{a}} e^{-\frac{k^2}{4a}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\beta^2 - \alpha^2} d\beta d\alpha$

$= \beta^2 + \alpha^2 = r^2$, $d\beta d\alpha = r dr d\theta$

$= \frac{\sqrt{\frac{a}{\pi}}}{\sqrt{a}} e^{-\frac{k^2}{4a}} \int_0^{2\pi} \int_0^{\infty} e^{-r^2} dr d\theta$

$= \frac{\sqrt{\frac{a}{\pi}}}{\sqrt{a}} e^{-\frac{k^2}{4a}} (\int_0^{2\pi} d\theta)(\int_0^{\infty} e^{-r^2} dr)$

let $u = r^2$, $du = 2rdr$, so $rdr = \frac{1}{2} du = dr$

$(\int_0^{2\pi} d\theta)(\frac{1}{2} \int_0^{\infty} e^{-u^2} du) = \pi$, but remember we squared the integral $\Rightarrow \sqrt{\pi}$

$= (\frac{\sqrt{\frac{a}{\pi}}}{\sqrt{a}} e^{-\frac{k^2}{4a}})(\sqrt{\pi})$

$\Rightarrow \frac{\sqrt{\pi}\sqrt{\frac{a}{\pi}}}{\sqrt{a}} e^{-\frac{k^2}{4a}} = e^{-\frac{k^2}{4a}}$

Note : the integration of $\sqrt{\frac{a}{\pi}} e^{-\beta^2 - \frac{k^2}{4a}}$ is possible because of contour integral techniques and because it is analytic and can be represented by a power series.

Frequency (υ) and time (t) dependent version of Fourier transform for use with optics :

$f(t) = \int_{-\infty}^{\infty} F(\upsilon)\, \mathrm{e}^{2\pi i \upsilon t} d\upsilon$

$F(\upsilon) = \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{-2\pi i \upsilon t} dt$

With the Fourier transform defined, the inverse seems to follow naturally. The inverse Fourier transform is actually the Fourier transform with positive exponent (Walker 160).

Definition of Inverse Fourier Transform

If f is continous and $\|f\|_1$ and $\|\hat{f}\|_1$ are both finite, then

$f(x) = \int_{-\infty}^{\infty} \hat{f}(u)\, e^{i2\pi ux} dx$ for all x $\in \mathbb{R}$ pg. 160

As aforementioned, convolution and Fourier transforms are interrelated in our context. However, with a slight modification to the definition of convolution, the relation is easily and noticeable and applicable. Be aware, that convolution is also referred to as "the folding product" and "composition product" by physicists (Steward 82).

Nevertheless, Fourier transforms have four very important properties which allow for simplification and useful manipulation in both physics and mathematics.
The properties are linearity, scaling, shifting, and modulation, which are defined below.

Linearity Theorem – For $\forall$a,b where a and b are constants the Fourier transform is said to have linearity if $af + bg \xrightarrow{\Im} a\hat{f} + b\hat{g}$ (Walker 155)

Scaling Theorem – $\exists p$, a positive constant $\ni$
$f(\frac{x}{p}) \xrightarrow{\Im} p\hat{f}(pu)$ and $f(px) \xrightarrow{\Im} \frac{1}{p}\hat{f}(\frac{u}{p})$

Modulation Theorem – $\exists$ a constant c $\in \mathbb{R}$ $\ni$
$f(x)e^{-i2\pi cu} \xrightarrow{\Im} \hat{f}(u-c)$

Shift Theorem – $\exists$ a constant c $\in \mathbb{R}$ $\ni$
$f(x-c) \xrightarrow{\Im} \hat{f}(u)e^{-i2\pi cu}$

References

http://www-history.mcs.st-andrews.ac.uk/history/Posters2/Fourier.html

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