Haar Wavelets, Image Compression, and Multi-Resolution Analysis

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Thesis: The purpose of this paper is to understand the implications of wavelet theory primarily through techniques of linear algebra with minimal emphasis on Fourier analysis. Much attention will be placed on the Haar wavelet (the simplest form of a wavelet) in one and two dimensions in order to understand its importance as a mathematical tool for hierarchically decomposing functions. Image compression and multi-resolution analysis will also be studied in order to fully understand wavelet applications.

Furthermore, on a more specific scale, given a function f, in W, we will identify elements in the nested subspaces that best approximate it in the L^2 norm. We are going to substitute the natural basis on W and its subspaces by one given by Haar wavelets. We will prove that in this new basis, the representation of f at various resolutions can be done by a very simple iteration process. Moreover, the reconstruction of the original function from the coefficients of this representation is equally simple and fast.

Introduction: Wavelets are at the forefront of both mathematics and engineering. By name, wavelets date back only to the 1980's and essentially provide an alternative to classical Fourier methods for both one and two dimensional data analysis and synthesis. Applications of wavelets are quite diverse and include astronomy, acoustics, nuclear engineering, sub-band coding, signal and image processing, neurophysiology, music, MRI, speech discrimination, optics, fractals, turbulence, earthquake prediction, radar, human vision, and solving partial differential equations. Ultimately, the emergence of wavelets compliments our fast paced, high speed, and information based lives.

A wave is usually referred to as an oscillating function of time and/or space. Fourier analysis is a type of wave analysis which expands signals in terms of sinusoids or complex exponentials. A wavelet, in a broad sense, is simply a small wave "with finite energy, which has its energy concentrated in time or space to give a tool for the analysis of transient, nonstationary, or time-varying phenomenon" (Reza 2). The wavelet still has the oscillating wavelike characteristics but it additionally has the ability to allow simultaneous time, or space, and frequency analysis.

The wavelet analysis procedure employs a wavelet prototype function, called mother wavelet. Temporal analysis is performed with a contracted high frequency version of the mother-wavelet, while frequency analysis is performed with a dilated, low

frequency version of the same wavelet. Because the original signal or function can be represented in terms of a wavelet expansion (using coefficients in a linear combination of the wavelet functions), data operations can be performed using just the corresponding wavelet coefficients. Smart choice of wavelet mother function results in wavelet basis yielding sparse data representation. In many cases wavelet coefficients get truncated below a threshold. This sparse coding makes wavelets an excellent tool for data compression. (http://www.smolensk.ru/user/sgma)

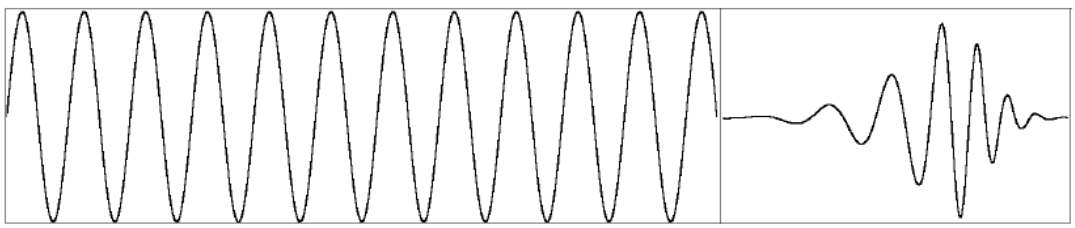


Figure 1- Comparison of a Wave and a Wavelet: Left graph is a Sine Wave with infinite energy and the right graph is a Wavelet with finite energy.

(Reza 2)

Before we hit the ground running, so to speak, with wavelets, it is imperative that a fundamental mathematical framework for wavelets be established first. The following will be a brief review of vector spaces, inner products and norms, and orthonormal bases, for each contribute greatly to wavelet theory. An inner product is a generalization of the dot product for vectors in $\Re^n$ and it gives a generalized notion of perpendicularity, called orthogonality. As we will see later on, orthogonal wavelets as well as localized basis are important in their contribution to data compression.

Recall that a vector space is a set V (whose elements are called vectors) equipped with two binary operations, called vector addition and scalar multiplication, for which certain properties hold. If x,y$\in$ V there is an element x+y in V called the vector sum of x and y. Also, if $\alpha \in \Re$ (the real numbers) and x$\in$ V there is an element α x in V, called the multiple of α and x. It is also important to note that if V is a vector space and we let $v_1, v_2, v_3, \ldots, v_n$ be elements of V we say that a linear combination of $v_1, v_2, v_3, \ldots, v_n$ is any vector of the form:

$\sum_{j=1}^{n} \alpha_j v_j = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$, where $\alpha_1, \alpha_2, \ldots \alpha_n \in R$ (Bartle 52).

There are many other properties that addition and scalar multiplication must satisfy for V to be a vector space.

If we let V be a vector space over $\Re$ (the real number field) then we use the notation $\langle\cdot,\cdot\rangle$ for a real inner product map $\langle\cdot,\cdot\rangle : \mathrm{VxV} \to \Re$. Recall that $\langle,\rangle$ has properties of additivity, scalar homogeneity, conjugate symmetry, and positive definiteness. A vector space V with an inner product is called an inner product space (Frazier 80).

Here are some examples of inner product space: L^2, where $\mathrm{L}^2 = \left\{ \{x_j\}_{j=1}^{\infty}, x_j \in \Re, \sum_{j=1}^{\infty} |x_j|^2 < \infty \right\}$, the set of all square-summable real sequences, and some of its subspaces L_N^2, where $\mathrm{L}_N^2 = \left\{ \{x_j\}_{j=1}^{2^N} : x_j \in \Re \right\}$. With the obvious component-wise addition and scalar multiplication, L^2 and L_N^2, are vector spaces over $\Re$. In applications such as signal processing, N is the number of resolutions for the wavelet functions.

The next two examples review topics such as norms and orthogonality (and what it means to be orthonormal). Letting V be the vector space of continuous functions on the interval [a,b] over the real numbers with inner product $\langle\cdot,\cdot\rangle$. For u,v$\in$ V we can define $\langle u,v\rangle = \int_a^b u(x)v(x)dx$ and $\|v\| = \sqrt{\langle v,v\rangle}$ where this norm generally agrees with the usual notion of the length of a vector, also known as the magnitude (Frazier 82). Some properties of the norm are as follows:

$\|f+g\| \le \|f\| + \|g\|$

$\|cf\| = |c|\|f\|$

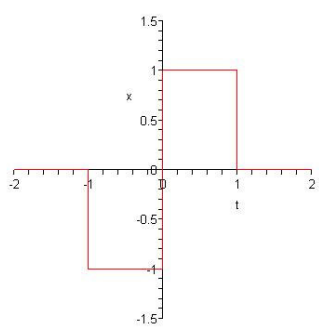
$\|f\| = 0 \Leftrightarrow f \equiv 0$

An extension of the norm is the distance between functions *f* and *g* which will come into play later when we try to find the error between two functions. By definition, the distance is $\|f-g\|$ and is denoted $d(f,g)$.

The definition of orthogonality is as follows. Suppose V is an inner product space. For *u*,*v*$\in$ V, we say that *u* and *v* are orthogonal (written $u \perp v$) if $\langle u,v\rangle$=0. Also, if we suppose that V is a real inner product space we can let B be a collection of vectors in V. B is an orthogonal set if any two different elements of B are orthogonal. B is an

orthonormal set if B is an orthogonal set and $\|v\|=1$ for all v$\in$ B. Orthogonal sets of nonzero vectors are linearly independent (Frazier 84).

It is also important to note that up to this point, continuous functions have been used. Discontinuous functions are a bit harder to work with but many wavelets, such as the Haar wavelet, are discontinuous so it is important to note the concept of "almost equal." Let *f* and *g* be piecewise continuous on [*a,b*]. We say that *f* and *g* are almost equal if there exists a finite set of points, *S*, such that for $x \in [a,b] \setminus S$ we have *f(x)=g(x)*. In the following case with the Haar wavelet, we are "ignoring" the points at x=-1,0,1. We will consider the set *S* the exception set. In summary, it is a fact that if *f* and *g* are almost equal on [*a,b*] and if x_0 is a point of continuity for both *f* and *g* then $f(x_0) = g(x_0)$.



(constructed from Heavyside function from maple)

The following is a lemma contingent on what has just been discussed. (almost equal=a.e.)

Lemma: If we let *f,g,h* be piecewise continuous on [*a,b*] such that *f=g* a.e. on [*a,b*] and *g=h* a.e. on [*a,b*], then:

(i) *f=h* a.e.

(ii) *f+g=g+h* a.e.

(iii) *f*g=f*h* a.e.

Proof of (i): By definition of *f=g* a.e. on [*a,b*] there exists a finite set S_1, such that for $x \in [a,b] \setminus S_1$, *f(x)=g(x)*. By definition of *g=h*, there exists a finite set S_2 such that for $x \in [a,b] \setminus s_2$, *g(x)=h(x)*. Now if we let $S_3 = S_1 \bigcup S_2$, it is considered to be a finite union since S_1, S_2 are finite. Let $x \in [a,b] \setminus S_3 \subset [a,b] \setminus S_2 \subset [a,b] \setminus S_1$. From this, *f(x)=g(x)* since $x \in [a,b] \setminus S_1$ and *g(x)=h(x)* since $x \in [a,b] \setminus S_2$. Therefore *f(x)=h(x)*. QED

The following is a lemma that will be used in our analysis of approximation and wavelet theory. Let f be a piecewise continuous on [a,b] and let V_0([a,b]) be the space of functions constant on [a,b] (meaning $g \in V_0$ iff $\exists c \in \Re$, g(x)=c for all x in [a,b]):

(i) Then there exists a unique function $f_0 \in V_0$ such that $\|f - f_0\| \le \|f - g\|$ for all $g \in V_0$

(ii) The function f_0 is equal to f(x)=average of f on [a,b] for all $x \in$ [a,b]=$\dfrac{\int_a^b f(t)dt}{b-a}$

The proof is as follows: Since function in V_0 are constant functions, all we need to show is that $\forall c \in \Re$ $\|f - f_{ave}\|^2 \le \|f - c\|^2$ where $f_{ave} = \dfrac{\int_a^b f(t)dt}{b-a}$. We thus want to show that $\int_a^b |f(t) - f_{ave}|^2 dt \le \int_a^b |f(t) - c|^2 dt$ (*). Consider the function $Q : \Re \to \Re$ defined by $Q(c) = \int_a^b |f(t) - c|^2 dt$. Note that the left side of (*) represents $Q(f_{ave})$. We will show it is a quadratic function, whose graph is a parabola. Its minimum is attained at $c = f_{ave}$ thus proving (*). Upon studying

$$Q(c) = \int_a^b (f^2(t) - 2f(t)c + c^2)dt = (\int_a^b f^2(t)dt) - 2c(\int_a^b f(t)dt) + c$$

So with this, $\dfrac{dQ}{dc} = -2\int_a^b f(t)dt + 2c(b-a)$. $\dfrac{dQ}{dc} = 0$ when $-2\int_a^b f(t)dt + 2c(b-a) = 0$ thus occurring when $c = \dfrac{\int_a^b f(t)dt}{b-a}$.

The previous theorem helps us approximate functions using the natural basis. The following is the main approximating theorem that allows us to replace the natural basis of

a function with the wavelet basis and from there we can determine the ever so important detail coefficients.

Theorem 1: If we let f be a piecewise continuous function on [-1,1] and we let V_n denote the space of functions piecewise constant on dyadic intervals of generation n:

(i) There exists $f_n \in V_n$ such that $\|f - f_n\| \le \|f - g\| \ \forall g \in V_n$

(ii) The function f_n is unique. It can be calculated as: for any dyadic interval of generation n, $J_n^k = \left[\frac{k}{2^{n-1}} - 1, \frac{k+1}{2^{n-1}} - 1\right]$, for k=0 to k= $2^n - 1$ and the restriction of f_n on J_n^k is equal to the average of f on J_n^k.

(iii) Reformulation of (ii): f_n can be calculated as $f_n = \sum_{k=1}^{2^n-1} c_k \chi_{J_n^k}$ then

$c_k = \frac{\int_{J_n^k} f_n dx}{length\ of\ J_n^k}$ where $\chi_{[a,b]} = 1\ for\ x \in [a,b]\ or\ 0\ for\ x \notin [a,b]$ (the characteristic function of $[a,b]$)

The proof is as follows. If we let g be an arbitrary function in V_n then we can use the additivity of integrals with respect to the interval of integration. Since [-1,1]= $J_n^0 \cup J_n^1 \cup J_n^2 \cup ... \cup J_n^{2^n-1}$ we get that

$$\|f - g\|^2 = \int_{[-1,1]} (f(x) - g(x))^2 dx = \sum_{k=0}^{2^n-1} \int_{J_n^k} (f(x) - g(x))^2 dx$$

If we use the previous lemma for f,g restricted to J_n^k. By g$\in V_n$, we get that g is constant on J_n^k. Now let c_n^k=$f_{ave,n}^k$ be the average of f on the interval J_n^k, thus

$c_n^k = \frac{1}{1/2^{n-1}} \int_{J_n^k} f(x)dx$. We get $\int_{J_n^k} (f(x) - c_n^k)^2 dx \le \int_{J_n^k} (f(x) - g(x))^2 dx$ (**) for

k=0,…,2^n-1, with equality if and only if $g\big|_{J_n^k}$ =constant c_n^k. Note that the function f_n described in Theorem 1 (ii) can be represented as $f_n = \sum_{k=0}^{2^n-1} c_n^k \chi_{J_n^k}$ and thus (**) can be written as $\int_{J_n^k} (f(x) - f_n(x))^2 dx \le \int_{J_n^k} (f(x) - g(x))^2 dx$ (***) for k=0,…,2^n-1, with equality

if $g\big|_{J_n^k} = f_n\big|_{J_n^k}$. Now add all of (***) with k=0,…,2^n -1 to get

$\sum_k \int_{J_n^k} (f(x) - f_n(x))^2 dx \le \sum_k \int_{J_n^k} (f(x) - g(x))^2 dx$ with equality if for all k=0,…,2^n -1

$g\big|_{J_n^k} = f_n\big|_{J_n^k}$ thus $\int_a^b (f(x) - f_n(x))^2 dx \le \int_a^b (f(x) - g(x))^2 dx \quad \forall g \in V_n$, with equality if and only if $g = f_n$. QED

As a note for the previous two proofs, their purpose is to simply show that there is a unique piecewise function that can approximate an original function better than any others.

Now that some of the basics are out of the way we turn our attention to wavelets using signal analysis as a mode of transition. When working in signal analysis, there are a number of different operations one can perform on that signal in order to translate it into different forms that are more suitable for different applications. The most popular function is the Fourier transform that converts a signal from time versus amplitude to frequency versus amplitude. This transform is useful for many applications, but it is not based in time. To combat this problem, mathematicians came up with the short term Fourier transform which can convert a signal to frequency versus time. Unfortunately, this transform also has its shortcomings mostly that it cannot get decent resolutions for both high and low frequencies at the same time.

The wavelet transform is a mechanism used to dissect or breakdown a signal into its constituent parts, thus enabling analysis of data in different frequency domains with each components resolution matched to its scale. Alternatively this may be seen as a decomposition of the signal into its set of basis functions (wavelets), analogous to the use of sines and cosines in Fourier analysis to represent other functions. These basis functions are obtained from dilations or contractions (scaling), and translations of the mother wavelet. The important difference that distinguishes the wavelet transform from Fourier analysis is its time and frequency localization properties. When analyzing signals of a non-stationary nature, it is often beneficial to be able to acquire a correlation between the time and frequency domains of a signal. In contrast to the Fourier transform, the wavelet transform allows exceptional localization in both the time domain via translations of the mother wavelet, and in the scale (frequency) domain via dilations.

Wavelets are finite windows through which the signal can be viewed. In order to move the window about the length of the signal, the wavelets can be translated about time in addition to being compressed, widened, or shifted.

Mathematically, a signal, or a raw image, is a function which is piecewise continuous on the interval [*a,b*]. We want to approximate it with simpler functions (piecewise constants in our case) such that the approximation is the best possible, within the designated class of simple functions. Yet as we extend the class of simple functions, the approximations can be improved. These extended classes of functions determine a succession of nested linear spaces $V_0 \subset V_1 \subset V_2 \ldots$. Ultimately, it comes down to the concept of "reusing" the part of the previous approximation to rebuild the new, better approximation. So in order to increase the resolution of an image while still using the previous base, a better basis needs to be established. The basis we are looking for is of the wavelet type.

The concept of high and low resolution from engineering, relates to the just mentioned nested linear spaces. If we start with the original space the original image or signal is at highest resolution. The image is approximated with functions in V_0 or V_1 etc. and is of low resolution and as we nest more vector spaces, the resolution continues to increase. This continues until one is satisfied with the compressed or approximated image or until there is no clear distinction between the raw image and what the wavelet basis has produced. Although the wavelet transform has come into prominence during the last decade, the founding principles behind wavelets can be traced back as far as 1909 when Alfred Haar discovered another orthonormal system of functions, such that for any continuous function f(x), the series

$$f(x) = \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} a_{2^j+k}\, w(2^j x - k), \qquad for \quad 0 \le x < 1$$

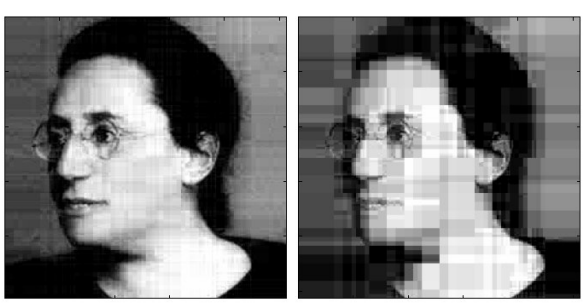
converges to f(x) uniformly over the interval $0 \le x < 1$.
(www.wavelet.org/tutorial/gifs/haar.gif) Here k represents the translation of the wavelet which is an indication of time and space. The integer k is also the center of the dyadic interval while j is referred to as the scale (dyadic in nature). W is the mother wavelet. So how can a signal be converted and manipulated while keeping resolution across the entire signal yet still be based in time? This is where wavelets come into play.

Aforementioned, the mathematical formulation of the signal-processing concept of high and low resolutions uses a sequence of nested subspaces of an inner product space of allowable functions (i.e. allowable signals). The functions we will be working with are either discrete signals or piecewise continuous functions, supported on the interval J=[-1,1]. Consider the dyadic subintervals of [-1,1], i.e. the intervals whose end points are of the form $\frac{k}{2^n}$, for integer k and positive integer n, listed below by their "generation:"

J_0=J, generation zero
J_1^0=[-1,0], J_1^1=[0,1], first generation,
J_2^0=[-1,-1/2], J_2^1=[-1/2,0], J_2^2=[0,1/2], J_2^3=[1/2,1], second generation
…
In general; $J_n^k=[\frac{k}{2^{n-1}}-1,\frac{k+1}{2^{n-1}}-1]$, where k=0 to k=$2^n$-1

For practical purposes one can think of an image as and element of a vector space such as V_j which would be the perfectly normal image, and the approximation of the original image with an element of V_{j-1} would be that image at a lower resolution, until you get to approximating the image with elements of V_0, where you just have one pixel in the entire image. Considering the spaces just mentioned, V_0, V_1, …, V_n, with functions defined on [-1,1] and piecewise constant on intervals of generations zero, generation one, generation n, respectively. In engineering language, these spaces are listed from low resolution to high resolution. These linear spaces have dimensions: $\dim V_0$=1, $\dim V_1$=2, $\dim V_2$=4,…,$\dim V_n$=2^n

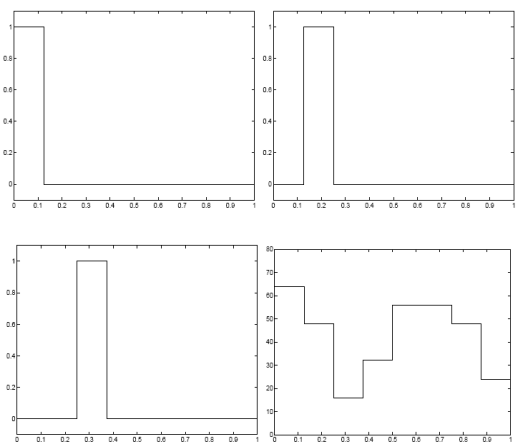
For each vector space V_j, there is an orthogonal complement called W_j and the basis function for this vector space is the wavelet.



(davis.wpi.edu)

And for each $0 \le i \le 2^3$-1, we get an induced (dyadically) dilated and translated scaling function: $\phi_i^3(x) = \phi(2^3 x - i/2^3)$. These eight functions form a basis for the vector space V^3 of piecewise constant functions on [0,1) with possible breaks at 1/8, 2/8, 3/8,…,7/8. It is important to note that ϕ_0^3 is 1 on [0,1/8) only, ϕ_1^3 is 1 on [1/8, 2/8) only, and so on. The following figure shows three of these basis functions together with a typical element of V^3. The last plot shows the unique element of,

$$64\phi_0^3 + 48\phi_1^3 + 16\phi_2^3 + 32\phi_3^3 + 56\phi_4^3 + 56\phi_5^3 + 48\phi_6^3 + 24\phi_7^3 \in \mathcal{V}^3,$$



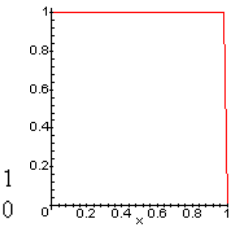
(Mulachy)

Now we make an introduction to the Haar wavelet. Alfred Haar (1885-1933) was a Hungarian mathematician who worked in analysis studying orthogonal systems of functions, partial differential equations, Chebyshev approximations and linear inequalities. Although the wavelets discussed in this paper had their origins in the early work of Haar, the subject has only really gathered momentum in the last decade. The Haar measure, Haar wavelet, and Haar transform are named in his honor.

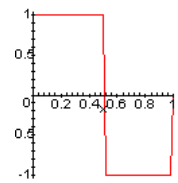
(O'Connor, Robertson 1)

The piecewise functions that make up the Haar wavelet and their graphs are as follows:

$$\phi := x \rightarrow \begin{cases} 0 \le x \text{ and } x < 1 & 1 \\ otherwise & 0 \end{cases}$$

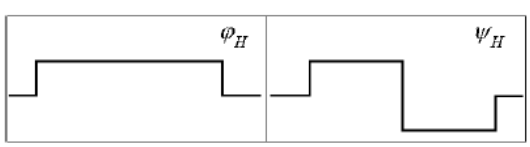


$$\psi := x \rightarrow \begin{cases} 0 \le x \text{ and } x < \frac{1}{2} & 1 \\ \frac{1}{2} \le x \text{ and } x < 1 & -1 \\ otherwise & 0 \end{cases}$$



(Mulachy)

Given a function f, in W, we have identified elements in the nested subspaces that best approximate it in the L^2 norm. We are going to substitute the natural basis on W and its subspaces by one given by Haar wavelets. In this new basis, the representation of *f* at various resolutions can be done by a very simple iteration process. The Haar wavelet can be considered to be a special type of step function that can in turn be thought of as a linear combination of dyadically dilated and translated unit step functions on [-1,1]. Note that the function ϕ satisfies a scaling equation of the form $\phi(x) = \sum_{i \in Z} c_i \phi(2x - i)$ where in this case the only nonzero c_i's are $c_0 = c_1 = 1$, i.e. $\phi(2x) + \phi(2x - 1)$.



Left graph is the Haar scaling function and the right one is the Haar mother wavelet.

(Reza 4)

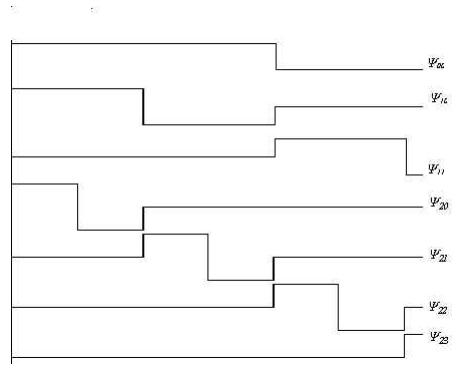


Fig.3. First 7 wavelets from Haar wavelet basis. (www.smolensk.ru/user/sgma)

Wavelet theory is based on analyzing signals to their components by using a set of basis functions. As seen above, one important characteristic of the wavelet basis functions is that they relate to each other by simple scaling and translation and that they recursively build upon eachother. The original wavelet function, known as the mother wavelet (i.e. the Haar mother wavelet), which is generally designed based on some desired characteristics associated to that function, is used to generate all basis functions. In most wavelet transform applications, it is required that the original signal be synthesized from the wavelet coefficients (the process of determining these coefficients will be discussed later). In general the goal of most modern wavelet research is to create a mother wavelet function that will give an "informative, efficient, and useful description of the signal of interest" (Reza 1). Some wavelets, simply based on their nature, are more suitable for particular applications than others. For example the Daubachy wavelet is suited for signals (such as noise signals) with sharp spikes and peaks and in some cases discontinuities.

As an extension, the wavelet transform is a two-parameter expansion of a signal in terms of a particular wavelet basis functions or wavelets. If we let ψ(t) represent the mother wavelet then all other wavelets are obtained by simple scaling and translation of ψ(t) as follows: $\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)$ where $2^{j/2}$ keeps the norm of the basis function equal to 1 (orthonormal property) i.e. $\frac{\psi}{\|\psi\|} = \frac{1}{1/\sqrt{2^j}}\psi = \sqrt{2^j}\psi = 2^{j/2}\psi$. The scaling is discrete and dyadic meaning a=2^{-j} and the translation is discretized with respect to each scale by using $\tau = k2^{-j}T$. In this case, the wavelet basis functions are obtained by

$\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - kT)$ where k,j$\in Z$ (integers) and T is the original size of the interval. The integer k represents translation of the wavelet function and is an indication of time or space in wavelet transform. Integer j, on the other hand, is an indication of the wavelet frequency or spectrum shift. It is referred to as the scale. The following are two different scaled versions of a wavelet along with the mother wavelet.

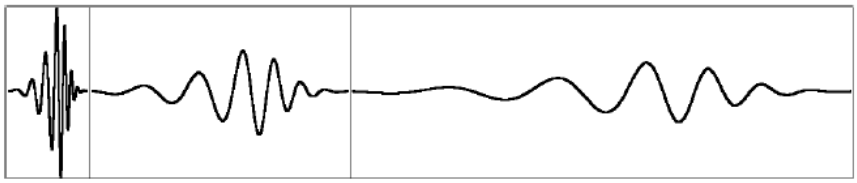


Figure 2- The left graph is the mother wavelet ψ_{D12}, the middle one is the wavelet at scale $j = -1$ and the right one is the wavelet at scale $j = -2$. The other way to look at these graphs is: to assume that the right graph is the mother wavelet, the middle one is the wavelet at scale $j = 1$ and the left one is the wavelet at scale $j = 2$.

(Reza 3)

Another aspect of the wavelet transform is that the localization or compactness of the wavelet increases as frequency or scale increases. In other words, higher scale corresponds to finer localization and vice versa (Reza 3). The paper titled *Wavelet Characteristics: What Wavelet should I Use*, written by Ali M. Reza provides elegant illustrations of the fundamentals of wavelets as well as multi-resolution analysis. She states that the multiresolution formulation needs two closely related basic functions. "In addition to the wavelet $\psi(t)$, there is a need for another basic function called the scaling function which is denoted $\phi(t)$." The two-parameter wavelet expansion for a signal designated as x(t) is given by the following decomposition series:

$$x(t) = c_k \phi_{j_0,k}(t) + \sum_k \sum_{n=0}^{N} d_{n,k} \psi_{n,k}(t)$$

The coefficients you see, c_k, are referred to as approximation coefficients at scale j_0, and the set of $d_{n,k}$ considered to be detail coefficients. The relationship of these wavelet coefficients to the original signal (for real and orthogonal wavelets) are given as:

$$d_{j,k} = \int x(t)\psi_{j,k}(t)dt$$
$$c_k = \int x(t)\phi_{j_0,k}(t)dt$$

Other examples of newly developed wavelets are shown in the following figure:

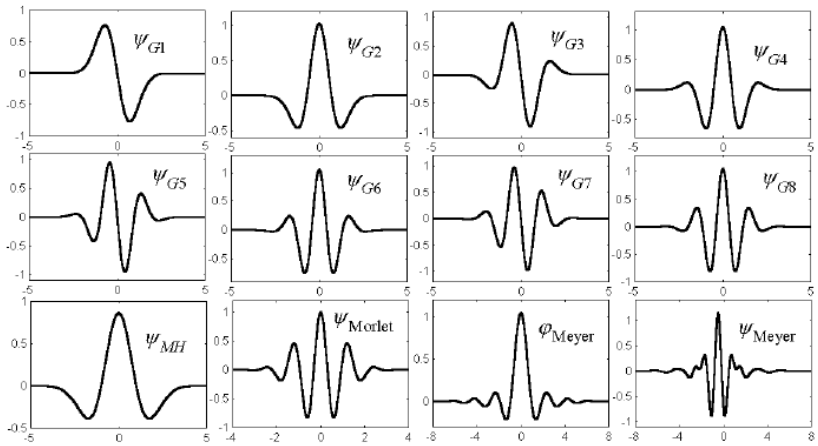
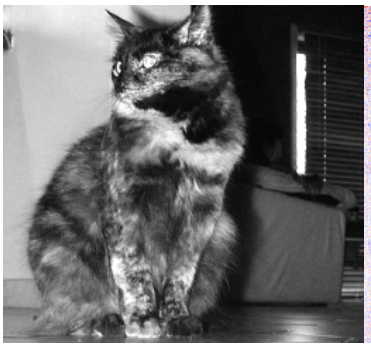
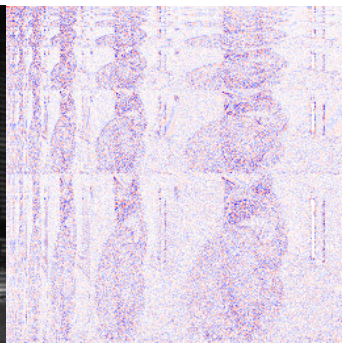


Figure 4- Different Gaussian wavelets obtained from derivatives of the Gaussian function along with Mexican Hat wavelet, Morlet wavelet and Meyer scaling function and wavelet. The order of the derivatives for Gaussian wavelets are shown as subscript for these wavelets.

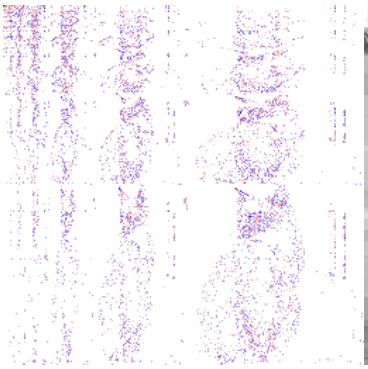
The following "images" of a cat show the fundamental transition between an original image and the final compressed image. It begins with substituting the normal basis for the image with the Haar wavelet basis. From this, we then find the coefficients that best approximate the original image through the means of the basis. Then the coefficients are quantized, or averaged. This is where the true concept of image compression comes into play. The goal is to drop or filter the largest amount of coefficients possible yet still be able to create a reconstructed image that is represented by fewer coefficients still similar to the raw image. The images show how the Haar transform can be used in image compression:



1. Original image



2. Haar coefficients



3. Quantized coefficients 4. Reconstructed image

An example of a practical application of the use of wavelet theory and analysis is seen in FBI fingerprint compression. Between 1924 and today, the US Federal Bureau of Investigation has collected about 30 million sets of fingerprints. Eventually, the need for a digitization and compression standard was needed. The fingerprint images are digitized at a resolution of 500 pixels per inch with 256 levels of gray-scale information per pixel. A single fingerprint itself is about 700,000 pixels and needs about .6 megabytes of storage while a pair of hands requires about 6 megabytes of storage. At a price of $900 per gigabyte for hard-disk storage and with about 200 terabytes of data to store, the FBI would have to pay about 200 million dollars to store uncompressed images. Application of wavelet analysis could bring this number way down (Graps11).

It is important to understand that wavelets are used in a series expansion of signals or functions much the same way a Fourier series used the wave or sinusoid to represent a signal or function. The big distinction between Fourier analysis and wavelet analysis is that for the Fourier series, sinusoids are chosen as basis functions, then the properties of the resulting expansion are examined. For wavelet analysis, one poses the desired properties and then derives the resulting basis functions (Burrus xi). Additionally, in wavelet analysis, the scale that is used to look at data is of particular importance. Wavelet algorithms process data at different scales or resolutions. If we look at a signal with a large "window," we would notice gross features. Similarly, if we look at a signal with a small "window," we would notice detailed features. The result in wavelet analysis is to see both the forest and the trees and be able to transfer this data quickly and efficiently. This is why wavelet analysis along with data or image compression is so unique and beneficial in our fast paced lives.

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