Wavelets and Image Compression

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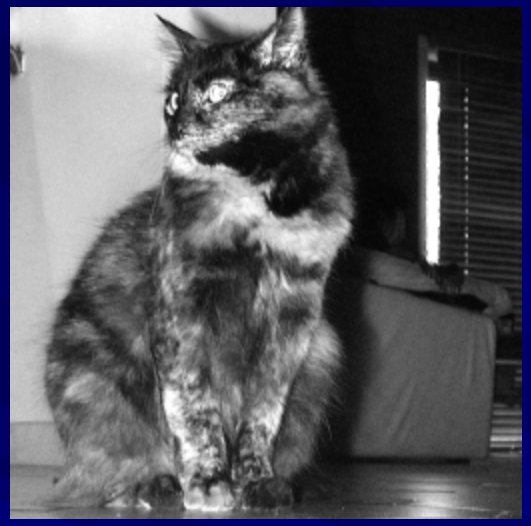
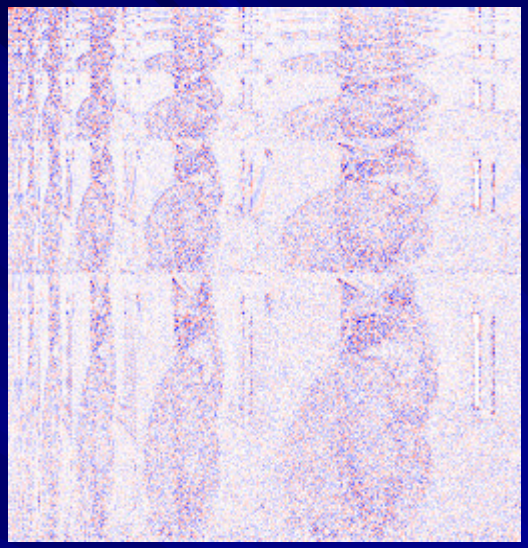
Capstone Project

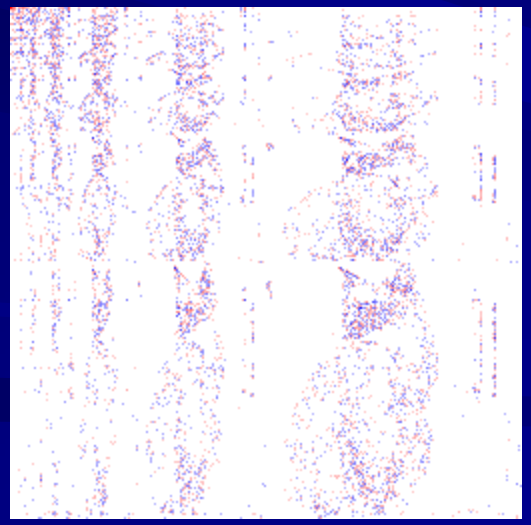
Goal: Achieve higher compression ratios while preserving reconstructed image quality



Original Image



Haar Coefficients



Quantized Coefficients

Reconstructed Image

A recent arrival on the mathematical scene…

- Gained momentum in the mid 1980's
  - much has stemmed from Haar's work in early 1900's.
- Emergence of computers called for an effective way to compress and transmit info (images and graphics, functions, signals, etc)
- …and an easy way to reconstruct original image

Historical Perspective

- Fourier's theory of frequency analysis: He asserted that any periodic function f(x) is the sum

$$a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

of its Fourier series. The coefficients are calculated by

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)dx, \quad a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(kx)dx, \quad b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(kx)dx$$

- Transition from frequency analysis to scale/resolution analysis

Albert Haar

- First mention of wavelets appeared in an appendix to the thesis of Haar (1909).
- Opened up the avenue for further wavelet and image compression study.

Wavelet theory

- Based on analyzing signals and their components using a set of basis functions.
  - Move out of analog (fxns) -> digital (vectors)
- It is also a tool for hierarchically decomposing functions: meaning it allows a function to be described in terms of a coarse overall shape, plus details that range from broad to narrow.

Vector Spaces

- A vector space is a set on which two operations, called (vector) addition and (scalar) multiplication occur. Must satisfy following axioms: associative, commutative, identity, inverse, distributive

$$V^0 \subset V^1 \subset V^2 \subset \cdots$$

- Consider V^j to be the perfectly normal image, and V^{j-1} the image at a lower resolution. Continue to V^0 where you just have one block for the entire image.

Dyadic Subintervals on [-1,1]

$J_0 = J$, generation zero

$J_1^0 = [-1,0]$, $J_1^1 = [0,1]$, first generation,

$J_2^0 = [-1,-1/2]$, $J_2^1 = [-1/2,0]$, $J_2^2 = [0,1/2]$, $J_2^3 = [1/2,1]$, second generation

…

In general; $J_n^k = [\frac{k}{2^{n-1}} - 1, \frac{k+1}{2^{n-1}} - 1]$, where k=0 to k=$2^n$ -1

This recursive property is not only seen in the intervals but within the coefficients as well. “Reuse” part of previous approximation to rebuild the new, better approximation.

L^2, Inner Product

- L^2 = the space of sequences $a_1, a_2, .. a_n$ in the real numbers such that the sum of a_n^2 is a convergent series.

- Let f,g be piecewise continuous on [a,b] then define the inner product of two functions as

$$< f, g >= \int_a^b f(x)g(x)dx$$

Properties of Wavelets

- Orthogonality: **inner product** of the mother wavelet with itself is unity.
- The inner products between the mother wavelet and its shifts and dilates of the mother are zero.
- The collection of shifted and dilated wavelet functions is called a wavelet **basis**.

Norm

- For a piecewise continuous function *f*

$$\|f\| = \sqrt{< f, f >} = \sqrt{\int_a^b f^2(x)dx}$$

$$\|f + g\| \le \|f\| + \|g\|$$
$$\|cf\| = |c| \, \|f\|, \text{ c=constant}$$
$$\|f\| = 0 \Leftrightarrow \text{f} \equiv 0$$

The distance between functions f,g by definition is: $\|f - g\| = \left[\int_a^b |f - g|^2 dt\right]^{1/2}$

The question: how do you find the “best” piecewise function g for a continuous function f?

Main Theorem on Approximations

- Let f be piecewise continuous on [-1,1]. Let V_n denote the space of functions piecewise constant on dyadic intervals of generation n.

(i) There exists $f_n \in V_n$ such that $\|f - f_n\| \le \|f - g\| \ \forall g \in V_n$

(ii) The function f_n is unique. It can be calculated as: for any dyadic interval of generation n, $J_n^k = \left[\frac{k}{2^{n-1}} - 1, \frac{k+1}{2^{n-1}} - 1\right]$, for k=0 to $k = 2^n - 1$ and the restriction of f_n on J_n^k is equal to the average of f on J_n^k.

(iii) Reformulation of (ii): f_n can be calculated as $f_n = \sum_{k=1}^{2^n-1} c_k \chi_{J_n^k}$ then

$$c_k = \frac{\int_{J_n^k} f_n dx}{length\ of\ J_n^k}$$ where $\chi_{[a,b]} = 1\ for\ x \in [a,b]\ or\ 0\ for\ x \notin [a,b]$ (the characteristic function of $[a,b]$)

Where I am going from here…

- Given a function f, in W, identify elements in the nested subspaces that best approximate it in the L^2 norm. Substitute the natural basis on W and its subspaces by one of the given Haar wavelets.
- Show that the reconstruction of the original function from the coefficients is equally simple and fast.

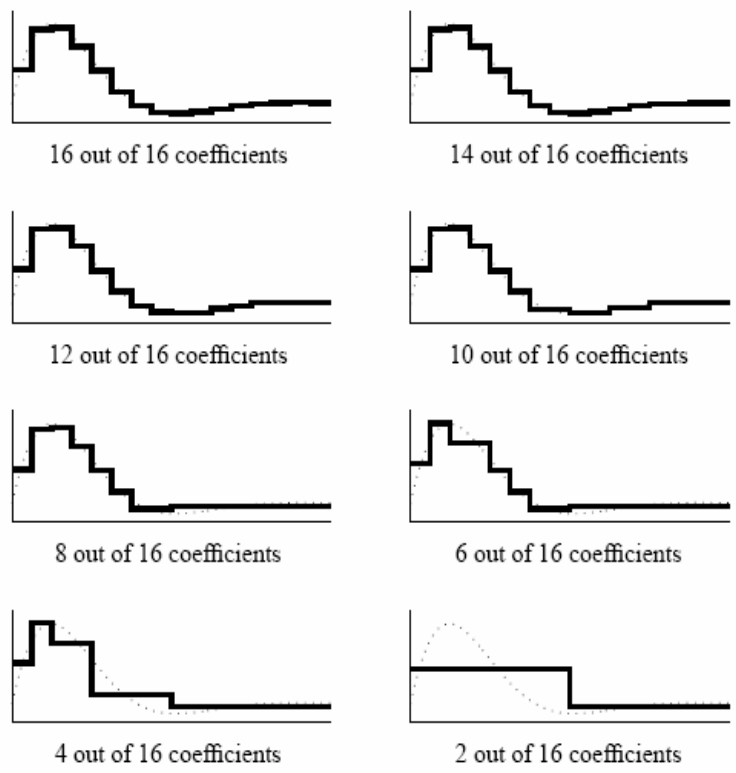
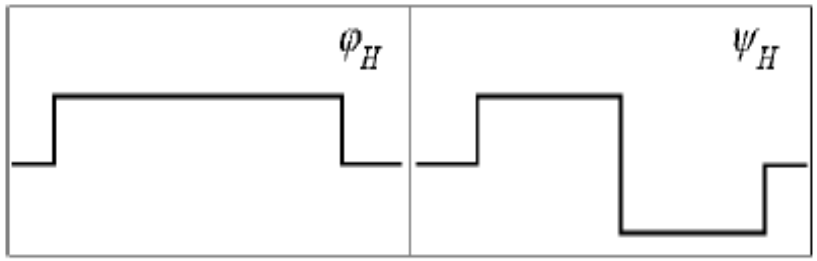


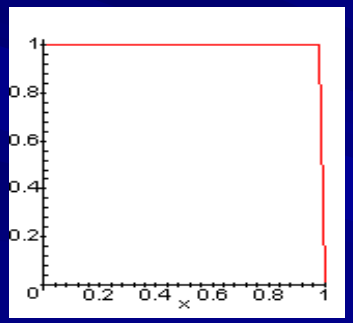
Figure 4 Coarse approximations to a function obtained using L^2 compression: detail coefficients are removed in order of increasing magnitude.

Introduction of the Haar wavelet.

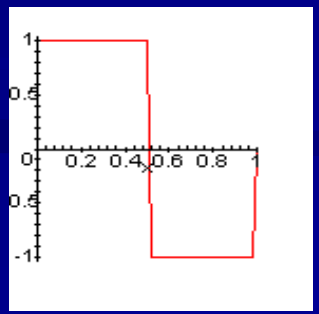


Left graph is the Haar scaling function and the right one is the Haar mother wavelet.

$$\phi := x \rightarrow \begin{cases} 1 & 0 \le x \text{ and } x < 1 \\ 0 & otherwise \end{cases}$$



$$\psi := x \rightarrow \begin{cases} 1 & 0 \le x \text{ and } x < \frac{1}{2} \\ -1 & \frac{1}{2} \le x \text{ and } x < 1 \\ 0 & otherwise \end{cases}$$



Let ψ (*t*) represent the mother wavelet. All other wavelets are obtained by simple *scaling* and *translation* of ψ (*t*) as follows:

$\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)$ where $2^{j/2}$ keeps the norm of the basis function=1 (orthonormal property)

i.e. $\frac{\psi}{\|\psi\|} = \frac{1}{1/\sqrt{2^j}}\psi = \sqrt{2^j}\psi = 2^{j/2}\psi$

k=translation

j=indication of wavelet freq shift; the scaling factor

Haar Wavelets

$$\psi_{j,k}(x) := \psi(2^j x - k) \text{ where } k = 0,\ldots,2^j - 1$$

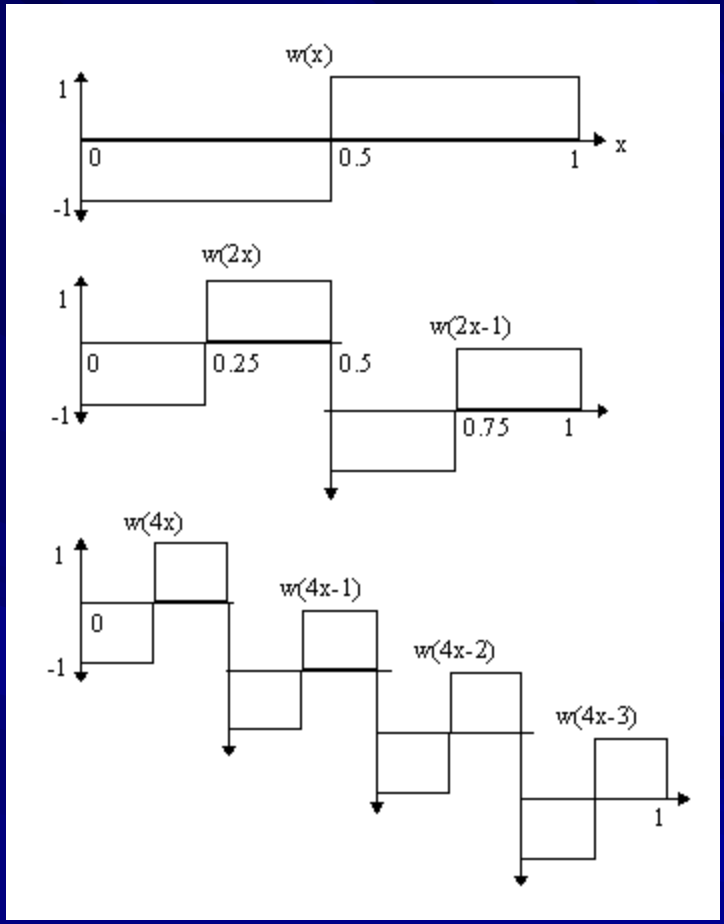
where

$$\psi(x) := \begin{cases} 1 \; for \; -1 \le x < 0 \\ -1 \; for \; 0 \le x < 1 \\ 0 \; otherwise \end{cases}$$

The original image I(x) can be expressed as a linear combination

$$I(x) = \underbrace{c_{2,0}\phi_{2,0}(x) + c_{2,1}\phi_{2,1}(x) + c_{2,2}\phi_{2,2}(x) + c_{2,3}\phi_{2,3}(x)}_{natural\ basis} = \overbrace{\underbrace{d_0\phi_0(x) + d_1\psi_{1,1}(x)}_{generation\ 1} + d_2\psi_{2,1} + d_3\psi_{2,2}}^{generation\ 2}$$

The “detail coefficients” are really coefficients of the wavelet basis.



Haar Wavelet, Levels 1, 2, 3

Mother Wavelet (The Prototype)

- Generally designed based on some desired characteristics associated to that function, is used to generate all basis functions
- All other wavelets are obtained by scaling and translation. Scaling is discrete, dyadic. $a = 2^{-j}$

$$\psi_{a,\tau}(t) = (1/\sqrt{a})\psi[(\frac{t-\tau}{a})] \qquad \tau = k2^{-j}T$$

  - Integer k: translation of wavelet fxn (indicates time/space)
  - Integer j: indication of scale (freq or spectrum shift)
  - T is the original size of the interval
- All other functions of the basis are variations of the mother wavelet

Two Different Scaled Versions of a Wavelet with the Mother Wavelet

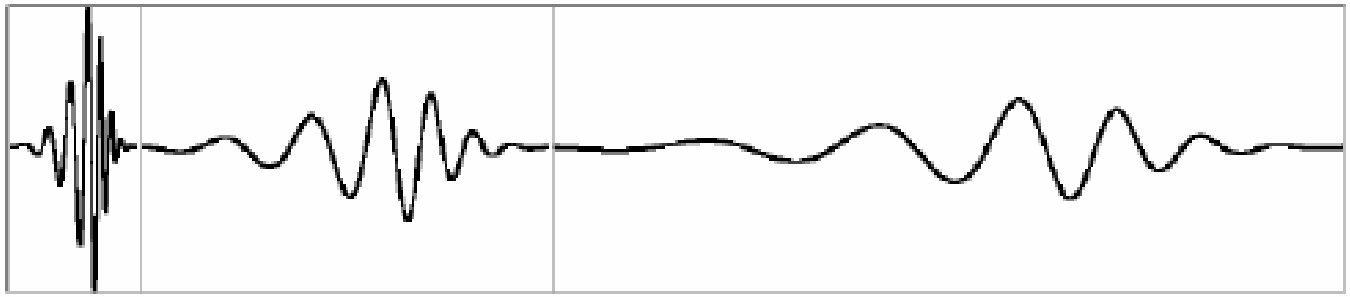


Figure 2- The left graph is the mother wavelet ψ_{D12}, the middle one is the wavelet at scale $j = -1$ and the right one is the wavelet at scale $j = -2$. The other way to look at these graphs is: to assume that the right graph is the mother wavelet, the middle one is the wavelet at scale $j = 1$ and the left one is the wavelet at scale $j = 2$.

Smart choice of wavelet mother function results in wavelet basis yielding sparse data representation.

Detail Coefficients

In addition to wavelet $\psi(t)$ there is a need for another basic function called the scaling function $\phi(t)$

Two parameter wavelet expansion for a signal designated as x(t) given by the

$$x(t) = c_k \phi_{j_0,k}(t) + \sum_k \sum_{n=0}^{N} d_{n,k} \Psi_{n,k}(t)$$

$$d_{j,k} = \int x(t) \Psi_{j,k}(t) dt$$

$$c_k = \int x(t) \phi_{j_0,k}(t) dt$$

"d" : referred to as detail coefficients
"c" : referred to as approximation coefficients at scale j_0

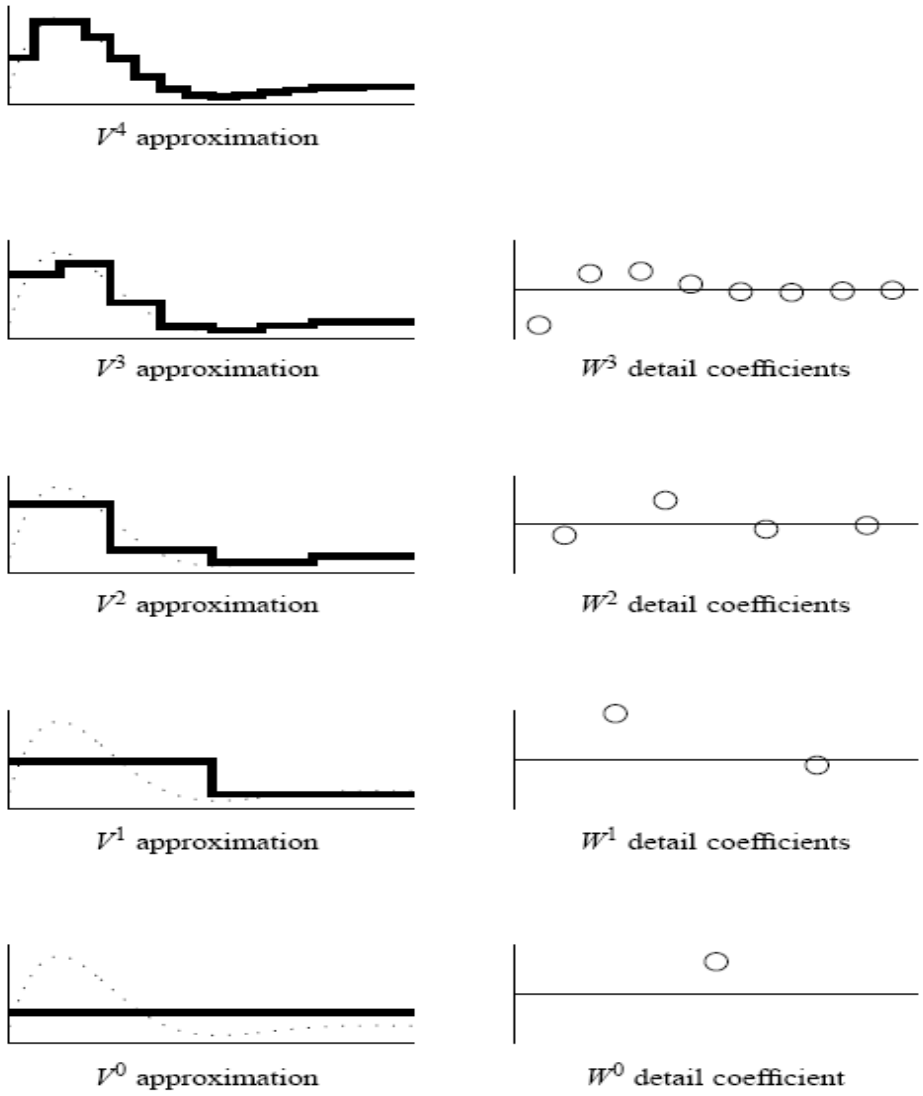


Figure 1 A sequence of decreasing-resolution approximations to a function (left), along with the detail coefficients required to recapture the finest approximation (right). Note that in regions where the true function is close to being flat, a piecewise-constant approximation works well, so the corresponding detail coefficients are relatively small.

Wavelet Classes

Orthogonal and orthonormal wavelet systems

Choosing the best wavelet will lead to the best approximation.

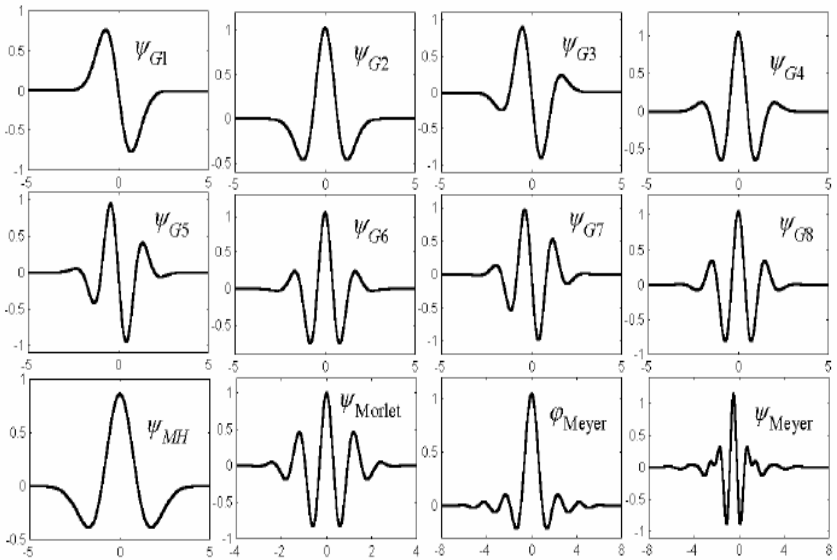


Figure 4- Different Gaussian wavelets obtained from derivatives of the Gaussian function along with Mexican Hat wavelet, Morlet wavelet and Meyer scaling function and wavelet. The order of the derivatives for Gaussian wavelets are shown as subscript for these wavelets.

Wavelet Transforms vs Fourier Transforms

- FFT: contains basis fxns that are sine and cosines
- Wavelet Transform: contains more complicated basis fxns called wavelets
- Coefficients are computed in analogous fashion (integrate product between the original fxn and the corresponding wavelet or cosine/sine fxn)
- But…
- Individual wavelets are functions localized in space (vanishing points)
- Can analyze physical situations where the signal contains discontinuities and sharp spikes
- Increase detail in a recursive manner

Applications

- Due to its simplicity and fast computational algorithm, Haar transforms are good for image processing.
- Because the wavelet transform can be applied to nonstationary signals and images, applications include:
- Nonlinear filtering or denoising, signal and image compression, speech coding, seismic and geological signal processing, medical and biomedical signal and image processing, and communication.

FBI Fingerprint Compression

- Between 1924-2007, 30 million sets of fingerprints have been collected.
- 1993, FBI's Criminal Justice Info Services Division developed standards for fingerprint digitization and compression.

Fig. 5. An FBI-digitized left thumb fingerprint. The image on the left is the original; the one on the right is reconstructed from a 26:1 compression. These images can be retrieved by anonymous FTP at `ftp.c3.lanl.gov` (128.165.21.64) in the directory `pub/WSQ/print_data`. (Courtesy Chris Brislawn, Los Alamos National Laboratory.

Data Storage Problem Continued…

- Fingerprint images are digitized at resolution; 500 pixels/inch w/ 256 levels of gray-scale info per pixel
- Fingerprint is 700,000 pixels -> .6Mbytes to store
- Pair of hands requires 6Mbytes…
- Results in 200 terabytes of data…($900/Gbyte for storage)
- Storage of uncompressed images = 200 million dollars !

Sources

- Bartle, Robert G., *The Elements of Real Analysis*, 2nd edition, John Wiley & Sons, New York NY, 1976.
- Burrus, C. Sidney, *Introduction to Wavelets and Wavelet Transforms: A Primer.* Prentice Hall, New Jersey. 1998
- Frazier, Michael W. *An Introduction to Wavelets Through Linear Algebra.* New York, 1991.
- Reza, Ali M., *Wavelet Characteristics: What Wavelet Should I Use?* http://www.xilinx.com/products/logicore/dsp/wavlet_char.pdf, October 19, 1999.
- Stollnitz, Eric J., Tony D. DeRose, David H. Salesin, Wavelets for Computer Graphics: A Primer. http://grail.cs.washington.edu/projects/wavelets/article/.