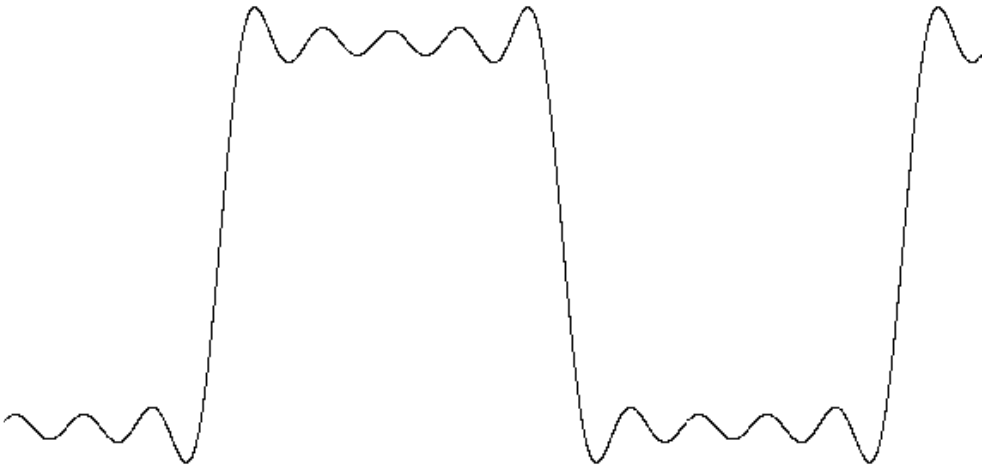
Filters and Fourier Transforms



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Initial Report

Capstone Project

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Jean Baptiste Joseph Fourier (1768-1830)

Jean Baptiste Joseph Fourier was born in March of 1768 in Auxerre, France. Fourier's mother died when he was only nine years old and his father died one year later. As a boy, Fourier studied at Pallais' school and then in 1780, he entered École Royale Militaire of Auxerre. As a student, Fourier was strong in Latin, French and Literature. By the age of 13, however, it was evident that mathematics was Fourier's true love. At 14, Fourier completed a study of the six volumes of Bézout's *Cours de Mathématiques*. Fourier was rewarded for his mathematical efforts with his first prize for his study of Bossut's *Mécanique en Général*.[1]

In 1787, Fourier entered the Benedictine abbey of St. Benoit-sur-Loire, to train to become a priest. While discerning his possible vocation in the religious life, he was torn because he knew that he was better suited as a mathematician, rather than a priest. While at seminary, Fourier wrote his math professor from Auxerre, "Yesterday was my 21st birthday. At that age, Newton and Pascal has already acquired many claims to immortality."[1] Not surprisingly, Fourier did not complete his training or take his religious vows.

After leaving St. Benoit's, Fourier became a math professor at École Royale Militaire of Auxerre, the same school he attended as a young man. In 1793, Fourier started his political career by joining his local Revolutionary Committee. Understandably, Fourier tried to resign from the committee after the French Revolutions' Reign of Terror. Since Fourier was heavily involved in the Revolution, he was not allowed to resign from his committee. Fourier spoke out against the committee, and was imprisoned in 1794. Fortunately, Fourier was released before heading to the guillotine.[1]

In 1794, Fourier was elected to study at École Normale in Paris—an institution focused on training teachers. At the École Normale, Fourier studied under Lagrange, Laplace, and Monge. Once his training was complete, Fourier taught at the Collège de France and then the École Polytechnique. At the École Polytechnique, Fourier rose to the position of Chair of Analysis and Mechanics and was known for his excellent lectures.[1]

Fourier joined Napoleon Bonaparte's army when invaded Cairo in 1798, and served as a scientific adviser. While in Egypt, Fourier helped found the Cairo Institute, serving as the Secretary during France's occupation. Fourier returned to France in 1801 and resumed his position as a math professor at École Polytechnique. Since Fourier was in Napoleon's good favor from their time together in Egypt, Napoleon appointed Fourier the Prefect in Grenoble. Although Fourier wanted to continue his work as a professor, he was in no position to refuse Napoleon.[1]

While in Grenoble, Fourier studied the theory of heat and wrote his famous work *On the Propagation of Heat in Solid Bodies* in 1807. Although this memoir is well-respected now, at the time, it was a very controversial piece. In 1808, Lagrange and Laplace objected that Fourier claimed he could expand functions as trigonometric series. This method is now known as Fourier series. Biot made a second objection as to how Fourier derived the heat transfer equations.[1] Laplace and Poisson has similar objections soon thereafter. Although Fourier's work was not favored, he received a prize in 1811 for his work on heat theory. Despite this reward, Fourier's work was not published due to its controversial nature.[1]

In 1817, Fourier was elected to the Académie des Sciences of Paris as the Secretary to the mathematical section.[1] Fourier wrote his published essay *Théorie*

Analytique de la Chaleur in 1822. During his last eight years, Fourier published several papers on pure and applied mathematics. Even after his death, Fourier's work inspired further study of trigonometric series and the theory of functions of a real variable.[1]

Fourier Series

The Fourier series for g is defined by the right side of the following correspondence:[2]

$$g \sim \sum_{n=-\infty}^{\infty} c_n \cdot e^{i2\pi nx/P} dx$$

If the function g has period P, then the Fourier coefficients $\{c_n\}$ for g are defined by:[2]

$$c_n = \frac{1}{P}\int_0^P g(x) \cdot e^{-i2\pi nx/P} dx$$

Discrete Fourier Transforms

DFTs are used in signal processing to analyze the frequencies contained in a sampled signal.

Given the N complex numbers, $\{h_j\}$ from j=0 to (N-1), their N-point DFT is denoted by $\{H_k\}$ where H_k is defined by: [2]

$$H_k = \sum_{j=0}^{N-1} h_j \cdot e^{-i2\pi jk/N}$$

Properties of DFTs:

Linearity - For all complex constants *a* and *b*, the sequence $\{ah_j + bg_j\}$ from j=0 to (N-1 has N-point DFT $\{aH_k + bGk\}$.[2]

Periodicity - For all integers *k* we have $H_k+N = H_k$. [2]

Inversion – For j = 0,1,…,N-1

$$h_j = \frac{1}{N}\sum_{j=0}^{N-1} H_k e^{i2\pi jk/N}$$

Fast Fourier Transforms

FFTs are an efficient algorithm to compute the DFT. Signal Processing uses FFTs to quickly multiply large integers. FFTs extract a set frequency domain from a signal, which creates a filter. [3]

Filters

A filter is a system which transmits or rejects a set range of frequencies.[3] Low-Pass Filter: passes low frequencies well, but reduces frequencies higher than the cut-off frequency High-Pass Filter: passes high frequency well, but attenuates frequencies lower than the cut-off frequency. Band-Pass Filter: attenuates very low and very high frequencies, but passes a middle range band of frequencies.

Ernesto Cesàro (1859-1806)

Cesàro Filter

With the Gibbs phenomena, the partial sums of the Fourier series seem to interlace around the graph of the step function. Since the functions oscillate from being above and below the step function, we find an average (or mean) to best approximate the function. The Cesàro filter is also known as the method of *arithmetic means*. [2]

Given a function with Fourier series partial sums $\{S_n\}$ from n=0 to ∞, the Mth arithmetic mean, or Cesàro filtered Fourier series using M harmonics, is denoted by σ_M where,[2]

$$\sigma_M = \frac{1}{M}[S_0 + S_1 + \ldots + S_{M-1}]$$

Cesàro filter

$$(1)\ \sigma_M = \frac{1}{M}[S_0 + S_1 + \ldots + S_{M-1}]$$

Replace the S's with a summation

$$(2)\ \sigma_M = \frac{1}{M}\sum_{k=0}^{M-1} S_k$$

Let $S_k = \sum_{j=-k}^{k} c_j \cdot e^{i\pi jx/L}$

$$(3)\ \sigma_M(x) = \frac{1}{M}\sum_{k=0}^{M-1}\left(\sum_{j=-k}^{k} c_j \cdot e^{i\pi jx/L}\right)$$

If we fix a value n, where n = 0,±1, ±2,…,±M, then we count how often C_n appears in the sums of (3) and obtain

$$(4)\ \sigma_M(x) = \sum_{n=-M}^{M}\left(1 - \left|\frac{n}{M}\right|\right) \cdot c_n \cdot e^{i\pi nx/L}$$

Compare σ_m with SM,

$$(4)\ \sigma_M(x) = \sum_{n=-M}^{M}\left(1 - \left|\frac{n}{M}\right|\right) \cdot c_n \cdot e^{i\pi nx/L}$$

$$(5)\ S_M(x) = \sum_{n=-M}^{M} c_n \cdot e^{i\pi nx/L}$$

We found SM by multiplying the coefficients in SM by the <u>filter factors</u>, $\left\{1 - \left|\frac{n}{M}\right|\right\}_{n=-M}^{n=M}$

Filter factors help improve the Cesàro filtered Fourier series to the original function and suppress the Gibbs phenomenon. [2]

<u>Butterworth Filter</u>

Works Cited

1. http://www-history.mcs.st-andrews.ac.uk/Biographies/Fourier.html

2. Walker, James, *Fast Fourier Transforms*, CRC Press, New York, 1996.

3. Lynn, Paul, *Digital Signal Processing*, Wolfgang Fuerst, New York, 1992.