## Introduction

When you hear a message broadcasted on the radio, the signal is crisp and clear despite the convoluted signal it plays. When you hear your voice on an answering machine, the voice sounds unfamiliar and different from the voice you hear when you speak because the telephone only transmits the lower frequencies of your voice. When you listen to a CD, the songs are rich, full-bodied sounds without interference despite that the CD is encrypted with a multitude of information. These are just a few examples of how filters improve a machine's performance. Digital signals are filtered using mathematical techniques called Fourier transforms and convolutions.

## Jean Baptiste Joseph Fourier (1768-1830)

Jean Baptiste Joseph Fourier was born in March of 1768 in Auxerre, France. Fourier's mother died when he was only nine years old, and his father died one year later. As a boy, Fourier studied at Pallais' school, and then in 1780, he entered École Royale Militaire of Auxerre. As a student, Fourier was strong in Latin, French and Literature. By the age of 13, however, it was evident that mathematics was Fourier's true love. At 14, Fourier completed a study of the six volumes of Bézout's *Cours de Mathématiques*. Fourier was rewarded for his mathematical efforts with his first prize for his study of Bossut's *Mécanique en Général*.[1]

In 1787, Fourier entered the Benedictine abbey of St. Benoit-sur-Loire to train to become a priest. While discerning his possible vocation in the religious life, he was torn because he knew that he was better suited to be a mathematician than to be a priest. While at seminary, Fourier wrote his math professor from Auxerre, "Yesterday was my

21st birthday. At that age, Newton and Pascal has already acquired many claims to immortality."[1] Not surprisingly, Fourier did not complete his training or take his religious vows.

After leaving St. Benoit's, Fourier became a math professor at École Royale Militaire of Auxerre, the same school he attended as a young man. In 1793, Fourier started his political career by joining his local Revolutionary Committee. Understandably, Fourier tried to resign from the committee after the French Revolution's Reign of Terror. Since Fourier was heavily involved in the Revolution, he was not allowed to resign from his committee. Fourier spoke out against the committee and was imprisoned in 1794. Fortunately, Fourier was released before heading to the guillotine.[1]

In 1794, Fourier was elected to study at École Normale in Paris—an institution focused on training teachers. At the École Normale, Fourier studied under Lagrange, Laplace, and Monge. Once his training was complete, Fourier taught at the Collège de France and then the École Polytechnique. At the École Polytechnique, Fourier rose to the position of Chair of Analysis and Mechanics and was known for his excellent lectures.[1]

Fourier joined Napoleon Bonaparte's army when they invaded Cairo in 1798 and served as a scientific adviser. While in Egypt, Fourier helped found the Cairo Institute, serving as the Secretary during France's occupation. Fourier returned to France in 1801 and resumed his position as a math professor at École Polytechnique. Since Fourier was in Napoleon's good favor from their time together in Egypt, Napoleon appointed Fourier the Prefect in Grenoble. Although Fourier wanted to continue his work as a professor, he was in no position to refuse Napoleon.[1]

While in Grenoble, Fourier studied the theory of heat and wrote his famous work *On the Propagation of Heat in Solid Bodies* in 1807. Although this memoir is well-respected now, at the time it was a very controversial piece. In 1808, Lagrange and Laplace objected that Fourier claimed he could expand functions as trigonometric series. This method is now known as Fourier series. Biot made a second objection as to how Fourier derived the heat transfer equations.[1] Laplace and Poisson made similar objections soon thereafter. Although Fourier's work was not favored, he received a prize in 1811 for his work on heat theory. Despite this reward, Fourier's work was not published due to its controversial nature.[1]

In 1817, Fourier was elected to the Académie des Sciences of Paris as the Secretary to the mathematical section.[1] Fourier wrote his published essay *Théorie Analytique de la Chaleur* in 1822. During his last eight years, Fourier published several papers on pure and applied mathematics. Even after his death, Fourier's work inspired further study of trigonometric series and the theory of functions of a real variable.[1]

## Discrete Fourier Transforms

Discrete Fourier Transforms (DFTs) are used to analyze and filter signals. First we will look at Fourier series and Fourier transforms, which are the basis of the DFT.

The Fourier series for $g$ is defined by the right side of the following correspondence: [2]

$$g \sim \sum_{n=-\infty}^{\infty} c_n \cdot e^{\frac{i \cdot 2 \cdot \pi \cdot n \cdot x}{P}}$$

Provided the function $g$ has period $P$ and the Fourier coefficients $\{c_n\}$ for $g$ are defined by:

$$c_n = \frac{1}{P}\int_0^P g(x)\cdot e^{\frac{-i\cdot 2\cdot\pi\cdot n\cdot x}{P}}dx$$

Given a function $f$ for which $\|f\|_1$ is finite, the Fourier transform is denoted by $\hat{f}$ and is defined as a function of $u$ by

$$\hat{f}(u) = \int_{-\infty}^{\infty} f(x)\cdot e^{-i\cdot 2\cdot\pi\cdot u\cdot x}dx$$

$$\text{where } \|f\|_1 = \int_{-\infty}^{\infty}|f(x)|dx$$

Given the N-complex numbers, $\{h_j\}_{j=0}^{N-1}$ their N-point DFT is denoted by $\{H_k\}$ where $H_k$ is defined by:

$$H_k = \sum_{j=0}^{N-1} h_j \cdot e^{\frac{-i\cdot 2\cdot\pi\cdot j\cdot k}{N}}$$

Specifically, DFTs are used in signal processing to analyze the frequencies contained in a sampled signal. For more information on Sampling, please refer to Midshipman Kelly Nelan's Capstone paper.

Properties of DFTs:

*Linearity* - For all complex constants $a$ and $b$, the sequence $\{ah_j + bg_j\}_{j=0}^{N-1}$ has N-point DFT $\{aH_k + bG_k\}$.

*Periodicity* - For all integers $k$ we have $H_{k+N} = H_k$.

*Inversion* – For $j = 0,1,\ldots,N-1$ $\quad h_j = \frac{1}{N}\sum_{k=0}^{N-1} H_k \cdot e^{\frac{i\cdot 2\cdot\pi\cdot j\cdot k}{N}}$

## Convolution Theorem

*Convolution Theorem*[2]: If $\{u_j\}$ and $\{v_j\}$ are sequences of period N with DFTs $\{U_k\}$ and $\{V_k\}$, respectively, then the Discrete Fourier Transform of the convolution of $u$ and $v_j$ is the product of $U_k$ and $V_k$; also written the DFT of $\{u * v_j\}$ is $\{U_k \cdot V_k\}$.

*Proof*: If we put $W = e^{\frac{-i \cdot 2 \cdot \pi}{N}}$, then the Discrete Fourier Transform of $\{u * v_j\}$ is

$$\sum_{j=0}^{N-1} u * v_j W^{j \cdot k} .$$

Replacing $u * v_j$ by the sum that defines it, we have upon rearranging sums

$$\sum_{j=0}^{N-1} u * v_j W^{j \cdot k} = \sum_{j=0}^{N-1} \left[ \sum_{m=0}^{N-1} u_m \cdot v_{j-m} \right] W^{j \cdot k}$$

$$= \sum_{m=0}^{N-1} u_m \left[ \sum_{j=0}^{N-1} v_{j-m} W^{j \cdot k} \right].$$

Replacing $W^{j \cdot k}$ by $W^{(j-m)k} W^{m \cdot k}$ in the last sum, yields

$$\sum_{j=0}^{N-1} u * v_j W^{j \cdot k} = \sum_{m=0}^{N-1} u_m W^{m \cdot k} \left[ \sum_{j=0}^{N-1} v_{j-m} W^{(j-m)k} \right]$$

Let dummy variable $p = (j - m)$

$$\sum_{j=0}^{N-1} u * v_j W^{j \cdot k} = \sum_{m=0}^{N-1} u_m W^{m \cdot k} \left[ \sum_{p=-m}^{N-m-1} v_p W^{p \cdot k} \right]. \qquad (1)$$

Now, the last sum in the brackets can be rewritten as follows:

$$\sum_{p=-m}^{N-m-1} v_p W^{p \cdot k} = \sum_{p=0}^{N-m-1} v_p W^{p \cdot k} + \sum_{p=-m}^{-1} v_p W^{p \cdot k}$$

$$= \sum_{p=0}^{N-m-1} v_p W^{p \cdot k} + \sum_{p=N-m}^{N-1} v_{p-N} W^{(p-N)k} .$$

Because $\{v_p\}$ has period *N*, we have $v_{p-N} = v_p$ for all *p*. Also, $W^{-N} = 1$. Therefore,

$$\sum_{p=-m}^{N-m-1} v_p W^{p \cdot k} = \sum_{p=0}^{N-1} v_p W^{p \cdot k} = V_k .$$

Hence, equation (1) becomes

$$\sum_{j=0}^{N-1} u * v_j W^{j \cdot k} = \sum_{m=0}^{N-1} u_m \cdot W^{m \cdot k} \cdot V_k = U_k \cdot V_k \qquad \blacksquare$$

Which proves that the Discrete Fourier Transform of $\{u * v_j\}$ is $\{U_k \cdot V_k\}$.

*Corollary* to the Convolution Theorem: The inverse DFT of $\{U_k \cdot V_k\}$ is $\{u * v_j\}$.

## **Filters**

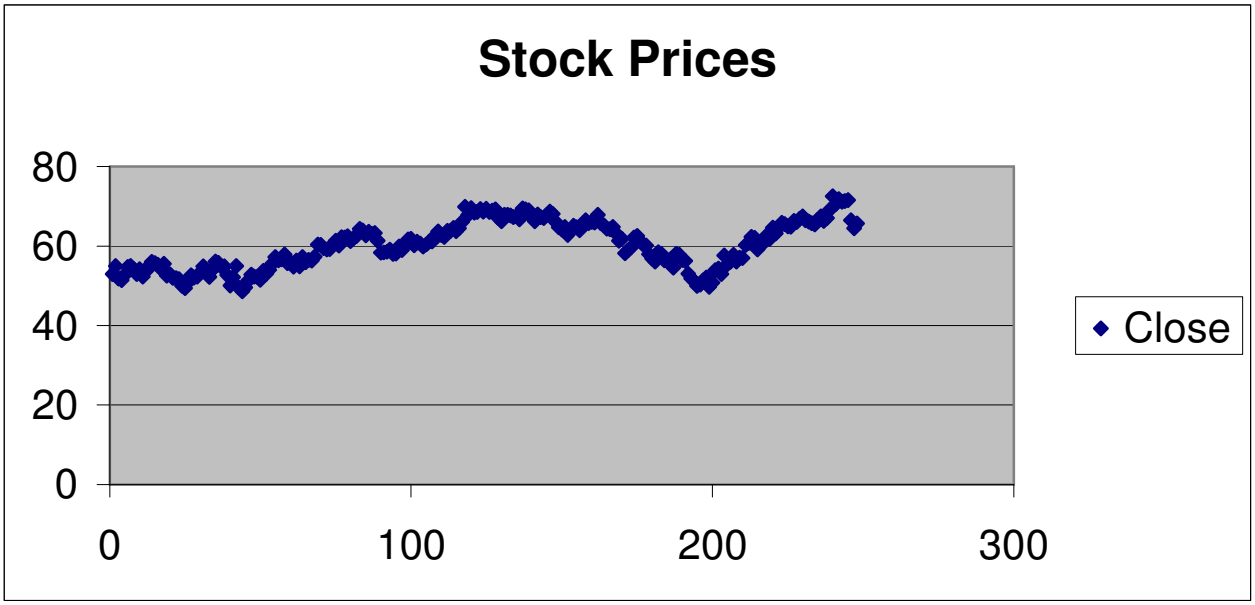
A filter is a system which transmits or rejects a set range of frequencies.[3] For example, when we listen to a certain signal, its low frequencies may come through clearly, but its high frequencies cloud the signal with irrelevant information, called "noise." When we reduce the noise, our signal transmits clearly without the unwanted interference. A filter reduces unnecessary frequencies to give us the best signal.[5]

There are three kinds of filters: low-pass filters, high-pass filters, and band-pass filters. Low-pass filters only pass low frequencies but reduce frequencies higher than the cut-off frequency. High-pass filters only pass high frequencies but reduce frequencies that are lower than the cut-off frequency. Band-pass filters reduce very low and very high frequencies but pass a middle range band of frequencies. The desired range of frequencies that the filter passes is called the passband.

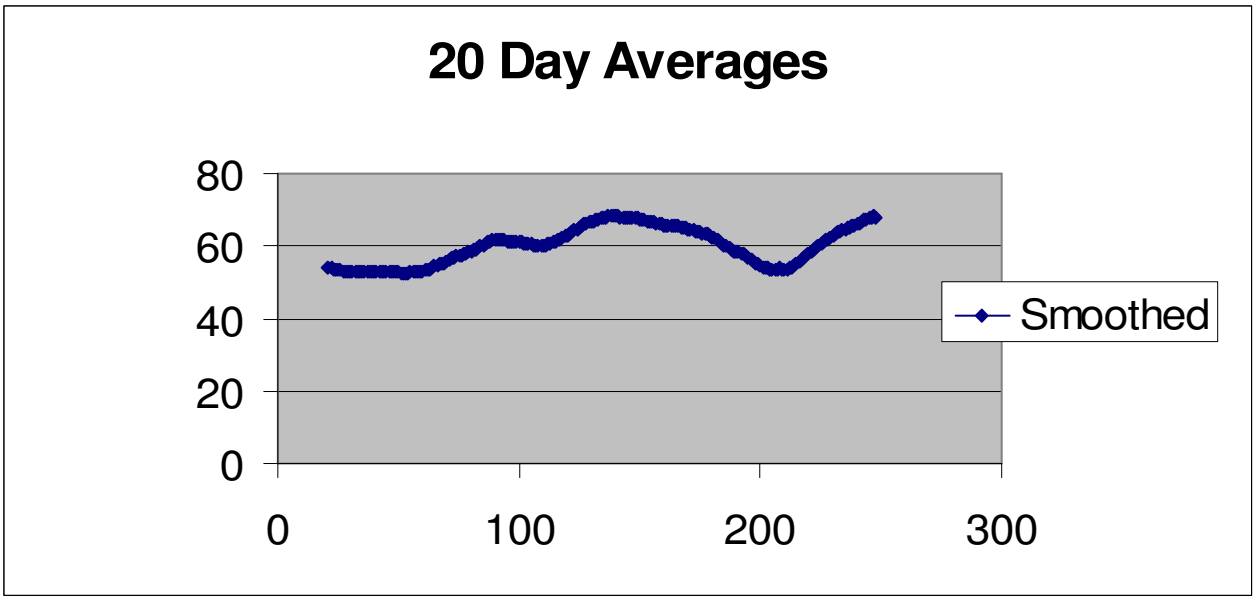
Filters are used in common applications such as subwoofers, radio transmitters, sound effects, CD players, virtual reality systems, and telephones.[4]

## Analyzing a Filter

To better understand filters, we will look at a specific example. The stock market is a continuously changing database. If we look at closing prices from one day at a time, we can analyze this data over a long period of time. The graph below shows the daily closing prices for a stock over a 300 day period. We analyze a filter of this data, which is be found by averaging the stock prices.




When comparing the two graphs, they have the same general shape. The graph below, however, is smoother than the graph above. It appears that we have a low-pass filter. Let us analyze this filter.




Let $y_n$ equal the output of our filter and $x_n$ equal the input. To find $y_n$, we will average the past 20 days of data. The point $x_n$ represents data from a single day.

The following equations define our filter:

$$y_n = \frac{1}{20} \cdot x_n + \frac{1}{20} \cdot x_{n-1} + \ldots + \frac{1}{20} \cdot x_{n-19}$$

$$y_n = b_0 \cdot x_n + b_1 \cdot x_{n-1} + \ldots + b_n \cdot x_0$$

where $b_0 = \frac{1}{20}, b_1 = \frac{1}{20}, \ldots, b_{19} = \frac{1}{20}, b_{20} = 0, \ldots$

Note that $y_n$ is the convolution of $x_n$ and $b_n$. Therefore, by the <u>convolution theorem</u>, its <u>Fourier transform</u> is defined as $Y_n = X_n \cdot B_n$.

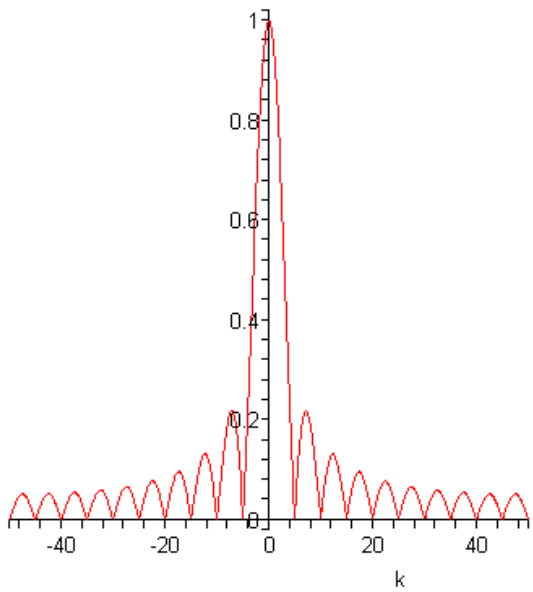
We have already defined the DFT as

$$B_j = \sum_{j=0}^{N-1} b_k \cdot e^{\frac{i \cdot 2 \cdot \pi \cdot j \cdot k}{N}}$$

In this example, let $N = 20$ and $b_k = \frac{1}{20}$.

Therefore, $$B_j = \sum_{j=0}^{19} \frac{1}{20} \cdot e^{\frac{i \cdot 2 \cdot \pi \cdot j \cdot k}{20}}$$

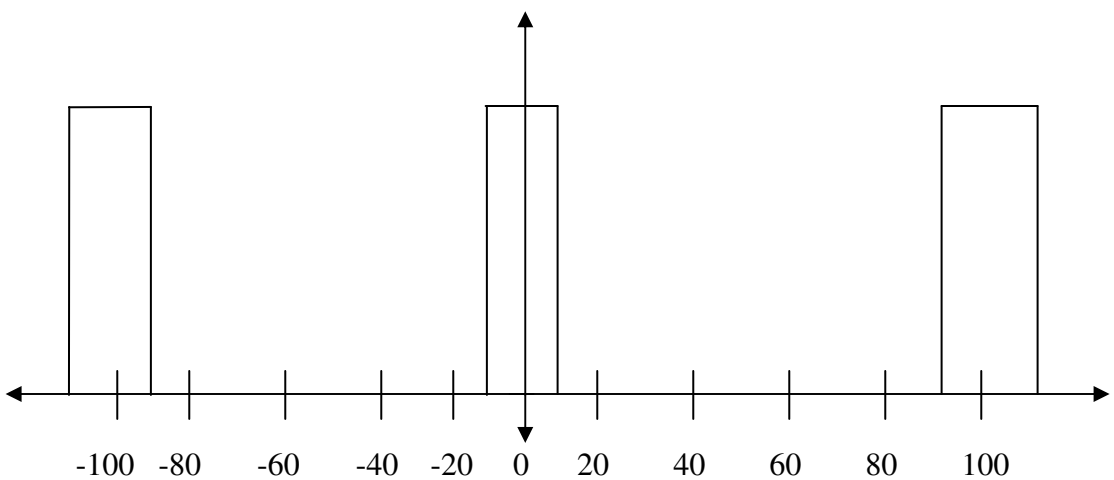
When we factor out the constants, $$B_j = \frac{1}{20} \sum_{j=0}^{19} e^{\frac{i \cdot 2 \cdot \pi \cdot j \cdot k}{20}}$$




Using the Maple program to produce a graph, we get the above figure. This is a graph of the $B_j$ s from our calculations. There is a spike on the graph where the frequencies are between $(-5,\ 5)$. From $(-50,\ -5)$ and $(5,\ 50)$ the signal is decreasing asymptotically to zero. This graph shows that $B_j$ lets the low frequencies $(-5,\ 5)$ pass, and the high frequencies $(-50,\ -5)$ and $(5,\ 50)$ are reduced to zero. Therefore, this graph represents a low-pass filter.

## Designing a Filter

Designing a filter is similar to analyzing a filter. When we analyzed the filter in the last example, we started with the $b_k$ values and found the Discrete Fourier Transform function $B_j$. Similarly, to design a filter, we will start with the desired $B_k$ function and use a DFT inversion to find the $b_j$ values.




The graph above represents our desired $B_k$ graph for a sample size of 100. This graph passes 20% of the signal and suppresses 80% of the signal. The portion of this graph from $(-10,\ 10)$ is part of the signal that is passed; the value here is 1. From $(10,\ 90)$ is where the signal is to be suppressed, so these values are 0. N represents one period of the signal; in this example, N = 100 since we will only analyze one period. Because we will use an inverse DFT, our N will go from [0, 99].

Therefore the desired $B_k$'s are:

$$B_k = \begin{cases} 1, & 0 \le k \le 9 \\ 0, & 10 \le k \le 89 \\ 1, & 90 \le k \le 99 \end{cases}$$

To find the $b_j$ values, we must use a DFT inversion. We have already defined the DFT inversion as:

$$b_j = \frac{1}{N}\sum_{k=0}^{N-1} B_k \cdot e^{\frac{i\cdot 2\cdot \pi\cdot j\cdot k}{N}}.$$

In our example, we will let N=100, and use the piecewise $B_k$ values to find $b_j$

$$b_j = \frac{1}{100}\sum_{k=0}^{9}(1)\cdot e^{\frac{i\cdot 2\cdot \pi\cdot j\cdot k}{100}} + \frac{1}{100}\sum_{k=10}^{89}(0)\cdot e^{\frac{i\cdot 2\cdot \pi\cdot j\cdot k}{100}} + \frac{1}{100}\sum_{k=90}^{99}(1)\cdot e^{\frac{i\cdot 2\cdot \pi\cdot j\cdot k}{100}}$$

Because $B_k$=0 between the values of k where $10 \le k \le 89$, the second summation goes to zero.

$$b_j = \frac{1}{100}\sum_{k=0}^{9} e^{\frac{i\cdot 2\cdot \pi\cdot j\cdot k}{100}} + \cancelto{0}{\frac{1}{100}\sum_{k=10}^{89} 0\cdot e^{\frac{i\cdot 2\cdot \pi\cdot j\cdot k}{100}}} + \frac{1}{100}\sum_{k=90}^{99} e^{\frac{i\cdot 2\cdot \pi\cdot j\cdot k}{100}}$$

Therefore, $$b_j = \frac{1}{100}\sum_{k=0}^{9} e^{\frac{i\cdot 2\cdot \pi\cdot j\cdot k}{100}} + \frac{1}{100}\sum_{k=90}^{99} e^{\frac{i\cdot 2\cdot \pi\cdot j\cdot k}{100}}$$

We will use Maple to compute the first six values of $b_j$:

$b_0 = 0.20$

$b_1 = 0.1870363000 - 0.00587785253\, i$

$b_2 = 0.1511661045 - 0.009510565\, i$

$b_3 = 0.1006112702 - 0.009510565\, i$

$b_4 = 0.0465279936 - 0.005877852\, i$

$b_5 = -\ 0.00000000025$

By analyzing the $b_j$ values, we can assume that since the imaginary values are so small, they are negligible and we will only use the real values. Also, since the value of $b_5$ is very small and all values of $b_n$ for $n \ge 5$ are even smaller, we will assume that all values

of $b_n$ for $n \geq 5$ are zero. Interestingly, if we compute the same $b_j$ values with N = 10,000, this larger sample results in approximately the same real parts, but even more minuscule imaginary parts.

This analysis proves that we only need to use the values $b_0$, $b_1$, $b_2$, $b_3$, and $b_4$ to write the function that represents this filter. In the example where we analyzed a filter, we knew that $y_n$ was the convolution of $x_n$ and $b_n$. Similarly when we design this filter, we must convolute the $x_n$ and $b_n$ values; the $y_n$ output is a convolution which is the function representing our filter:

$$y_n = b_0 \cdot x_n + b_1 \cdot x_{n-1} + b_2 \cdot x_{n-2} + b_3 \cdot x_{n-3} + b_4 \cdot x_{n-4} + \text{(the zero terms)}$$

Ideally, when you add the $b_j$ values in the $y_n$ equation, their sum should equal 1. We are only using the values $b_0$, $b_1$, $b_2$, $b_3$, and $b_4$ since they had significant values, but their sum does not equal 1. To best approximate the $y_n$ function we must <u>normalize</u> the $b_j$ values. To normalize these values, we will find the actual sum of the $b_j$ values, and divide each $b_j$ by its sum.

$$b_0 + b_1 + b_2 + b_3 + b_4 \cong 0.685$$

When we divide each bj value by its normalizing number, we get

$$y_n = \frac{b_0}{0.685} \cdot x_n + \frac{b_1}{0.685} \cdot x_{n-1} + \frac{b_2}{0.685} \cdot x_{n-2} + \frac{b_3}{0.685} \cdot x_{n-3} + \frac{b_4}{0.685} \cdot x_{n-4}$$

Substituting the computed $b_j$ values into this equation gives the function

$$y_n = \frac{0.20}{0.685} \cdot x_n + \frac{0.1870}{0.685} \cdot x_{n-1} + \frac{0.1512}{0.685} \cdot x_{n-2} + \frac{0.1006}{0.685} \cdot x_{n-3} + \frac{0.0465}{0.685} \cdot x_{n-4}$$

$$y_n = 0.2918 \cdot x_n + 0.2730 \cdot x_{n-1} + 0.2207 \cdot x_{n-2} + 0.1469 \cdot x_{n-3} + 0.0679 \cdot x_{n-4}$$

This $y_n$ equation represents the output of our filter in a similar way to the averaging low-pass filter on Page 8.

## Famous Filters

The two filters above are generic and simple compared to more complex, famous filters. Several mathematicians have left their mark by designing a filter that performs a specific task. For example, some filters use a method of arithmetic means to approximate the best signal, while other filters best approximate a step-function through a series of harmonics. We will look at two famous filters: the Cesàro filter and the de la Vallée Poussin filter.

## Cesàro Filter

The Cesàro filter is also known as the method of *arithmetic means*.[2] Given a function with Fourier series partial sums $\{S_n\}_{n=0}^{\infty}$, the M$^{th}$ arithmetic mean, or Cesàro filtered Fourier series using M harmonics, is denoted by $\sigma_M$ where,

$$\sigma_M = \frac{1}{M}[S_0 + S_1 + \ldots + S_{M-1}]$$

Cesàro filter

$$\sigma_M = \frac{1}{M}[S_0 + S_1 + \ldots + S_{M-1}] \qquad (1)$$

Replace the $S$ values with a summation

$$\sigma_M = \frac{1}{M}\sum_{k=0}^{M-1} S_k \qquad (2)$$

Let $\quad S_k = \sum_{j=-k}^{k} c_j \cdot e^{\frac{i\pi jx}{L}}$

$$\sigma_M(x) = \frac{1}{M}\sum_{k=0}^{M-1}\left(\sum_{j=-k}^{k} c_j \cdot e^{\frac{i\cdot\pi\cdot j\cdot x}{L}}\right) \qquad (3)$$

If we fix a value n, where n = 0, ±1, ±2,…, ±M, then we count how often

$C_n$ appears in the sums of (3) and obtain

$$\sigma_M(x) = \sum_{n=-M}^{M} \left(1 - \left|\frac{n}{M}\right|\right) \cdot c_n \cdot e^{\frac{i \cdot \pi \cdot n \cdot x}{L}} \qquad (4)$$

Compare $\sigma_m$ with $S_M$,

$$\sigma_M(x) = \sum_{n=-M}^{M} \left(1 - \left|\frac{n}{M}\right|\right) \cdot c_n \cdot e^{\frac{i \cdot \pi \cdot n \cdot x}{L}} \qquad (5)$$

$$S_M(x) = \sum_{n=-M}^{M} c_n \cdot e^{\frac{i \cdot \pi \cdot n \cdot x}{L}} \qquad (6)$$

We found $S_M$ by multiplying the coefficients in $S_M$ by the filter factors, $\left\{1 - \left|\frac{n}{M}\right|\right\}_{n=-M}^{n=M}$

The Gibbs phenomenon is a situation where the partial sums of the Fourier series seem to interlace around the graph of the step function. Since the functions oscillate from being above and below the step function, we find an average (or mean) to best approximate the function.

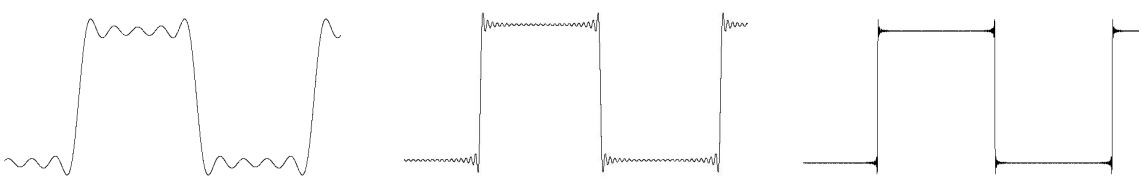
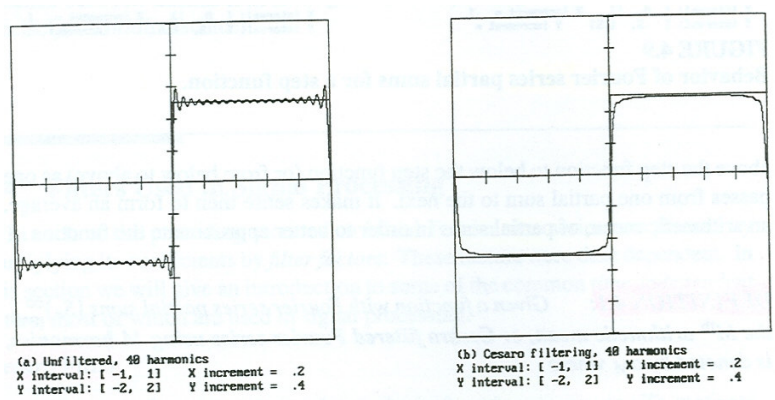


Figure 1 Figure 2 Figure 3

Figures 1, 2, and 3 show a signal experiencing the Gibbs Phenomenon. The shape of the filtered signal is nearly that of the desired step function, but will never be a smooth step function.




The figure above shows the Cesàro filter suppressing the Gibbs phenomenon. [2]

## de la Vallée Poussin filter

The de la Vallée Poussin (dlVP) filter is very similar to the Cesàro filter, but the dlVP filter uses only the upper half of partial sums to find the best approximation for a step function.[2]

If we have an even number of harmonics, 2M, then we define $V_{2M}$ by

$$V_{2M} = \frac{1}{M} \sum_{k=M}^{2M-1} S_k \ . \qquad (1)$$

This equation is called the dlVP filtered partial sum using 2M harmonics.[2]

We can now show that

$$V_{2M}(x) = \sum_{n=-2M}^{2M} v_n \cdot c_n \cdot e^{\frac{i \cdot \pi \cdot n \cdot x}{L}} \qquad (2)$$

$$\text{where} \quad v_n = \begin{cases} 1, & |n| \le M \\ 2 \cdot \left(1 - \left|\frac{n}{2 \cdot M}\right|\right), & M < |n| \le 2 \cdot M \end{cases} \qquad (3)$$

First, we rewrite (1) by using the definition of $S_k$:

$$V_{2M}(x) = \frac{1}{M}\sum_{k=M}^{2M-1}\left[\sum_{j=-k}^{k} c_j \cdot e^{\frac{i\cdot\pi\cdot j\cdot x}{L}}\right]. \qquad (4)$$

Second, we must find a coefficient of $e^{\frac{i\cdot\pi\cdot n\cdot x}{L}}$ for each n = 0, ±1,…, ±2M. We msut consider two cases.

*Case 1*: ($|n| \le M$). In this case, $c_n$ appears in each term in brackets in (4) for

$k$ = M to (2M -1). Therefore, the exponential $e^{\frac{i\cdot\pi\cdot n\cdot x}{L}}$ has coefficient

$$\frac{1}{M}\sum_{k=M}^{2M-1} c_n = c_n\left[\frac{1}{M}\sum_{k=M}^{2M-1} 1\right] = c_n$$

Therefore $v_n$ = 1 as in (3).

*Case 2*: ($m < |n| \le 2M$). For this case $c_n$ appears in only those terms in the brackets in (4)

where k = $|n|$,…, (2M-1). Therefore, the coefficient of $e^{\frac{i\cdot\pi\cdot n\cdot x}{L}}$ is

$$\frac{1}{M}\sum_{k=|n|}^{2M-1} c_n = c_n\left[\frac{1}{M}\sum_{k=|n|}^{2M-1} 1\right] = c_n \cdot \frac{2M-|n|}{M}$$

$$= \left(2 - \left|\frac{n}{M}\right|\right)\cdot c_n = 2\cdot\left(1 - \left|\frac{n}{2M}\right|\right)\cdot c_n$$

Therefore, $v_n = 2\cdot\left(1 - \left|\frac{n}{2M}\right|\right)$ as in (3).

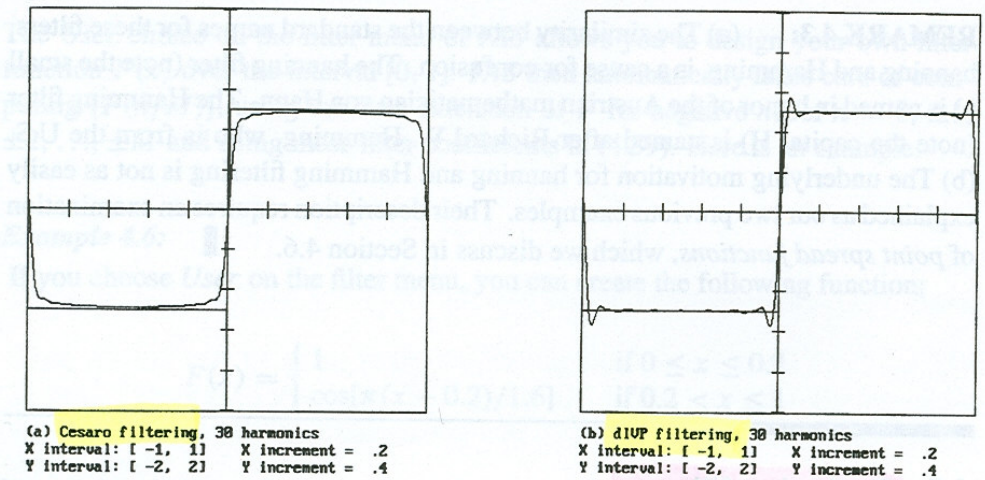



**FIGURE 4.12**

In Figure 4.12, we compare a Cesàro filter and a de la Vallée Poussin filter.[2] We graphed a dlVP filtered Fourier series partial sum for a step function. By analyzing the graphs we can see that dlVP filtering gives a closer approximation to the original step function than the Cesàro filter.

## Conclusion

When tasked to write a Capstone paper, the thought of writing a minimum of 15 pages about mathematics was daunting, to say the least. Choosing one of the seven possible topics relating to Fourier series was like playing mathematical Russian roulette; I had no previous knowledge about any of my topic choices and I could only hope to choose a topic that I would comprehend. Thankfully, "Filters and Fourier Transforms" turned out to have real-life applications that I understood and I was able to learn about Fourier series and Fourier transforms in the process. I enjoyed analyzing and designing filters with my Capstone advisor, Professor Richard Maruszewski, and I was able to associate the concepts of Fourier transforms to my filter examples.

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4. http://en.wikipedia.org/wiki/Audio_filter

5. Briggs, William, *The DFT*, SIAM, Philadelphia, 1995.