Applications of Fourier Transforms in Solving Differential Equations
Draft Report for Capstone Project

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Introduction

Mathematicians have long had techniques to solve quadratic equations, geometric questions, and for physical conditions of speed or position. But, since Pythagoras, the study of the vibrating string, important due to a continuous interest in music, was hampered by a lack of appropriate techniques. Solutions to the one-dimensional wave equation did not come about until the eighteenth century. Those solutions conflicted with one another until Joseph Fourier successfully reconciled them in his solution to the thermodynamic heat equation in the next century. Since that time, Fourier series have been a successful technique to solving equations with separable variables, including Erwin Schrödinger's equation describing the space-time relationship in quantum mechanics.

History

Music has always been an important part of human existence. From the beginning, people were interested in the scales and in the vibration of strings of instruments. However, exploring the mechanics and mathematics of the strings proved impossible for centuries. It was not until after partial derivatives came into use that Jean le Rond d'Alembert, Leonard Euler, and Joseph-Louis Lagrange were able to arrive at solutions to the one-dimensional wave equation in the eighteenth-century. Daniel Bernoulli also obtained a solution, but as it dealt with series of sines and cosines, his contemporaries did not believe that he could be correct. Bernoulli's technique would be later repeated by Joseph Fourier as he solved the thermodynamic heat equation in 1807.[1]

Daniel Bernoulli was born into a family of mathematicians, though his father did not want him to study mathematics. He did so anyway, among other subjects, and obtained his master's degree at the age of 16 in Philosophy. This dispute over his career was not the only conflict between Daniel and his father, Johann Bernoulli. Johann has been said to steal material from his son's most famous work, *Hydrodynamica* (1738), prior to its publication and republish it as his own in *Hydraulica* (c. 1739) in an attempt to show that his son had in fact plagiarized from him.[2] He finished medical school in 1720, and used his math skills (that he had been taught on the side during his education) to write his doctoral thesis on the mechanics of breathing. He worked with Leonhard Euler from 1727 until 1733 in St. Petersburg. In St. Petersburg, he showed that "the movements of strings of musical instruments are composed of an infinite number of harmonic vibrations all superimposed on the string."[3] He died in 1782 in Basel, Switzerland, the city where he held the chairs of metaphysics, mathematics, and natural philosophy at the University of Basel.

Jean le Rond d'Alembert was the illegitimate son of Louis-Camus Destouches, who arranged for his education. He attended Jansenist Collège des Quatre Nations, and over the next few years studied law, mathematics, and medicine. He joined the Paris Academy of Science in 1741, based on his work on fluid mechanics and integral calculus. He was a contemporary of Bernoulli, and in fact in his paper, *Traité de l'équilibre et du mouvement des fluids*, gave an "alternative treatment of fluids" that Bernoulli did not agree with.[4] He solved the one dimensional wave equation when given "arbitrary" data, which was published in 1747. D'Alembert, Euler, and Bernoulli all did work after that to determine just how arbitrary it could be.[5] Bernoulli's method involved the separation of variables in solving the differential equation, which would be the technique later employed by Fourier. D'Alembert is also known for finding the Cauchy-Riemann Equations years before they were determined by Cauchy or Rieman. He died in 1783 in Paris, France of a bladder illness, and was buried in an unmarked grave.[6]

Jean Baptiste Joseph Fourier, a prominent French mathematician, was the first to solve the heat equation while he was in Egypt. While at the École Normale in Paris, he was taught by famous mathematicians, like Lagrange, Laplace, and Monge, then taught at the Collège de France. Like his contemporaries, he studied many subjects besides mathematics, including literature, and at one point he was studying for the priesthood. In 1798, he joined Napoleon's Army and went to Egypt, where he helped found the Cairo Institute, before returning to Paris in 1801. His work on the heat equation was completed in 1807 and presented as *On the Propagation of Heat in Solid Bodies* to the Paris Institute, where it caused several controversies: Lagrange and Laplace were not convinced by his explanations of his "expansions of functions as trigonometrical series, what we now call Fourier series."[7] Also, Jean-Baptiste Biot (and later Laplace and Poisson) was unhappy with Fourier's derivations of the heat equation. In 1817, Fourier was elected to the Académie des Sciences, and in 1822 was made the Secretary of the mathematics section.[8] Shortly thereafter, his most famous work, *Théorie analytique de la chaleur*, was published.

Erwin Rudolf Josef Alexander Schrödinger was born in Vienna, Austria in 1887, and was educated by a private tutor until the age of ten. At the University of Vienna, he studied analytical mechanics, applications of partial differential equations to dynamics, eigenvalue problems, Maxwell's equations and electromagnetic theory, optics, thermodynamics, and statistical mechanics, and received his doctorate in 1910 with the paper *On the conduction of electricity on the surface of insulators in moist air.*[9] He served in World War I, though he did continue his research while on the Italian front. His “revolutionary work relating wave mechanics and the general theory relativity” was published in 1926, and was the “second formulation of quantum theory” after Heisenberg, and those six papers were well received by his contemporaries, including Albert Einstein.[10] His famous cat paradox was published in 1935 as *The present situation in quantum mechanics*. He spent the latter years of the 1930’s conflicting with and fleeing from the new Nazi regime in Austria. In Dublin starting in 1939, he and Einstein worked together on the unified theory, though in 1947 their partnership was ended after Schrödinger published a paper that Einstein did not agree with. In addition to physics, he was also interested in biology, and published the book *What is Life?* in 1944.[11] He died in Vienna in January 1961 of tuberculosis. For his work in quantum mechanics and physics, a crater on the dark side of the moon was named after him.

Concepts Used

Fourier Coefficients

The Fourier Coefficients C_n can be found using the following formula:

$$C_n = \frac{1}{P}\int_0^P f(x) e^{\frac{-2\pi i n x}{P}} dx$$

Fourier Series

The basic Fourier Series is given by:

$$FS(f)(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{2\pi i n x}{P}}$$

Superposition Principle

The Superposition Principle states that for a linear system, the sum of solutions for that system is also a solution.

Wave Equation

$$\boxed{\frac{\partial^2 u}{\partial t^2} = \omega^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{Boundary conditions}: \quad u(0,t) = 0 \qquad u(L,t) = 0 \qquad \text{Initial Conditions}: \quad u(x,0) = f(x) \quad \frac{\partial u}{dt}(x,0) = g(x)}$$

The wave equation is normally studied in the context of a vibrating string, as in a musical instrument. Jean le Rond d'Alembert, an 18th Century French mathematician, derived the general solution to the one dimensional scalar wave equation, d'Alembert's Formula: $u(x,t) = \dfrac{f(x-ct) + f(x+ct)}{2} + \dfrac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$ using the above conditions.[12]

An early version of the Fourier solution was given by Daniel Bernoulli, using techniques previously used by Leonhard Euler, prior to Fourier's own work.

Solution to Wave Equation

We start by assuming that the solution to the wave equation is separable.

$u(x,t) = X(x)T(t)$

$$\frac{\partial u}{\partial t} = XT' \qquad \frac{\partial^2 u}{\partial t^2} = XT''$$

$$\frac{\partial u}{\partial x} = X'T \quad \frac{\partial^2 u}{\partial x^2} = X''T$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial t^2} = \omega^2 \frac{\partial^2 u}{\partial x^2} \quad \Rightarrow \quad T''X = \omega^2 X''T$$

By Separation of Variables we get :

$$\frac{T''}{\omega^2 T} = \frac{X''}{X} = -\lambda$$

The only way the two sides can be equal is if they are both equal to a constant, - λ

To Solve for X : $X'' = -\lambda X \quad \Rightarrow \quad X'' + \lambda X = 0$

$(D^2 + \lambda)X = 0 \quad \Rightarrow \quad D = \pm\sqrt{\lambda}i$

$\Rightarrow \quad X = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$

Using the boundary conditions we get :

$X(0) = C_1 = 0$ because $\cos(0) = 1$ and $\sin(0) = 0$

We assume $\lambda > 0$ because by the boundary conditions, $\exists C_1, C_2 \ni X(x) = C_1 e^{-\sqrt{-\lambda x}} + C_2 e^{-\sqrt{\lambda x}}$

if $\lambda < 0$, $X(0) = 0 = X(L) \Rightarrow C_1 = 0 = C_2 \Rightarrow u = 0$

Similarly for $\lambda = 0$, $\exists C_1, C_2 \ni X(x) = C_1 x + C_2$. By the boundary conditions, $u = 0$. $\Rightarrow \lambda > 0$

$X(L) = C_2 \sin\sqrt{\lambda}L = 0 \quad \Rightarrow \quad \sin\sqrt{\lambda}L = 0$ when $\sqrt{\lambda}L = n\pi \Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2$ n

$$X = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right) \quad \text{where } C_1 = 0$$

To Solve for T : $T'' = -\omega^2\lambda T \quad \Rightarrow \quad T'' + \omega^2\lambda T = 0$

$(D^2 + \omega^2\lambda)T = 0 \quad \Rightarrow \quad D = \pm\left(\frac{\omega n\pi}{L}\right)i = \pm\gamma_n i \quad n \in \mathbb{Z}$

Therefore : $T = D_1 \cos\gamma_n t + D_2 \sin\gamma_n t$

Thus we get : $u(x,t) = X(x)T(t)$

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right)\left(D_n \cos\gamma_n t + E_n \sin\gamma_n t\right) \text{ by Superposition Principle}$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left(D_n \cos\gamma_n t + E_n \sin\gamma_n t\right)$$

Using the Initial Conditions : $u(x,0) = f(x)$

$$u(x,0) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left(D_n \cos\gamma_n 0 + E_n \sin\gamma_n 0\right) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$$

where $D_n = \frac{2}{L}\int_0^L f(x)\sin\sqrt{\lambda_n}\,x\,dx$

$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} \gamma_n \sin\left(\frac{n\pi}{L}x\right)\left(-D_n \sin\gamma_n t + E_n \cos\gamma_n t\right)$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \gamma_n \sin\left(\frac{n\pi}{L}x\right)\left(-D_n \sin\gamma_n 0 + E_n \cos\gamma_n 0\right) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi}{L}x\right) = g(x)$$

where $E_n = \frac{2}{\gamma_n L}\int_0^L g(x)\sin\sqrt{\lambda_n}\,x\,dx$

Final Solution

$$\boxed{\begin{array}{l} u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left(D_n \cos\frac{\omega n\pi}{L}t + E_n \sin\frac{\omega n\pi}{L}t\right) \\ \text{where } D_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx \text{ and } E_n = \frac{2}{\gamma_n L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx \end{array}}$$

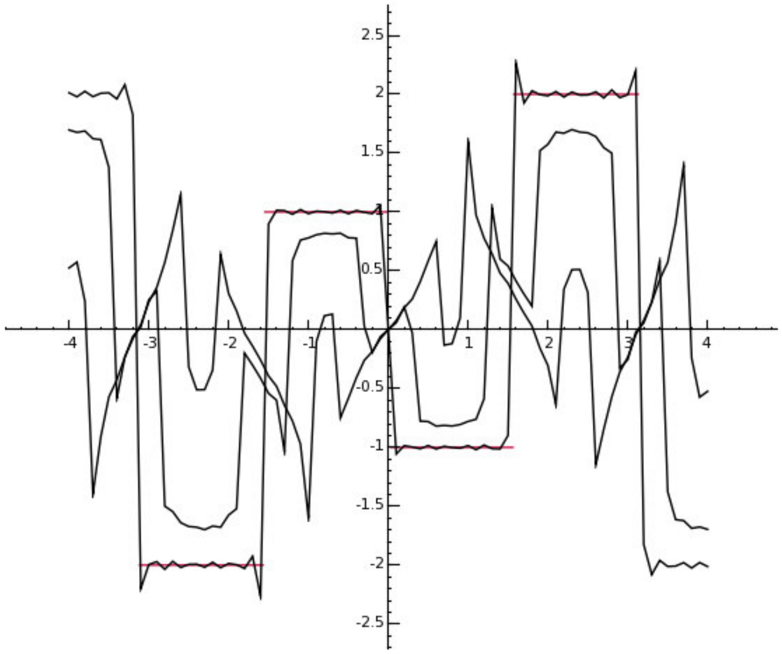

Figure 1: Wave Equation un-filtered

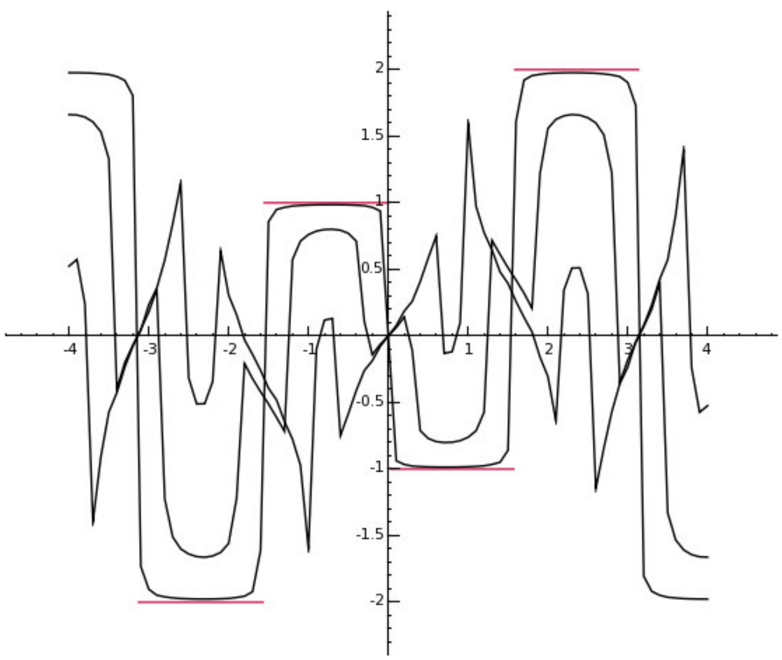

Figure 2: Wave Equation using Cesaro Filter

Heat Equation (with zero-endpoints)

$$\frac{\partial u}{\partial t} = \mathrm{k}\frac{\partial^2 u}{\partial \mathrm{x}^2} \qquad \text{Boundary Conditions}: u(0,t) = 0 \qquad u(L,t) = 0$$

$$\text{Initial Conditions}: \quad u(x,0) = f(x)$$

As has already been stated, Joseph Fourier first solved the heat equation and published his work in his *Théorie analytique de la chaleur* in 1822. From his work on this problem, he claimed that any function on the interval $0<x<L$ could be expanded into a sine series. Though this has since been proved false, Fourier sine series have proved useful in solving partial differential equations.

Solution to Heat Equation

Assume that the solution is separable. Thus:

$$u(x,t) = X(x)T(t)$$

$$\frac{\partial u}{\partial t} = XT' + X'T$$ Because we are taking the derivative with respect to t the $X'T$ will go to 0

$\Rightarrow$

$$\frac{\partial u}{\partial t} = XT'$$

$$\frac{\partial u}{\partial x} = X'T \Rightarrow \qquad \frac{\partial^2 u}{\partial x^2} = X''T \quad \text{similarly}$$

From these partial differential equations we get the following equation :

$$\frac{\partial \mathrm{u}}{\partial \mathrm{t}} = k\frac{\partial^2 u}{\partial x^2} \quad \Rightarrow \; XT' = kX''T$$

Using seperation of variables we get :

$$\frac{\mathrm{T'}}{kT} = \frac{X''}{X} = -\lambda \quad \text{Similar to the wave equation}$$

Now to solve for x : $X'' = -\lambda X \quad \Rightarrow \quad X'' + \lambda X = 0$

$(D^2 + \lambda)X = 0 \;\; \Rightarrow \;\; D = \pm\sqrt{\lambda}i$

$\Rightarrow \; X = A_1 \cos\sqrt{\lambda}x + A_2 \sin\sqrt{\lambda}x$

$\in$ Z

Using the boundary conditions, we get the same result as in the Wave Equation :

$$X = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) \quad \text{where } \mathrm{A}_1 = 0$$

Now to solve for T : $T' = -\lambda kT \;\; \Rightarrow \;\; T' + \lambda kT = 0$

$(D + \lambda k)T = 0 \;\; \Rightarrow \;\; D = -\lambda k$

$\Rightarrow \;\; T = Be^{-\lambda kt}$

Thus, we get : $u(x,t) = X(x)T(t)$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\sqrt{\lambda_n}x(Be^{-\lambda_n kt}) = \sum_{n=1}^{\infty} A_n \sin\sqrt{\lambda_n}xe^{-\lambda_n kt}$$

Using the initial condition : $u(x,0) = f(x)$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\sqrt{\lambda_n}x = f(x)$$

$$\Rightarrow \;\; A_n = \frac{2}{L}\int_0^L f(x)\sin\sqrt{\lambda_n}xdx$$

Final Solution:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n \sin\sqrt{\lambda_n}\, x e^{-\lambda_n k t} \quad \text{where} \quad A_n = \frac{2}{L}\int_0^L f(x)\sin\sqrt{\lambda_n}\, x dx}$$

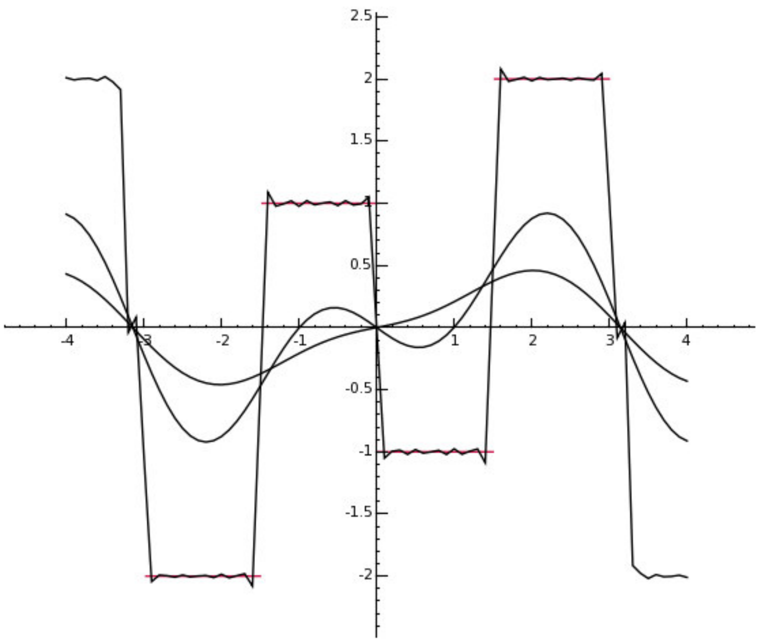

Figure 3: Heat Equation un-filtered

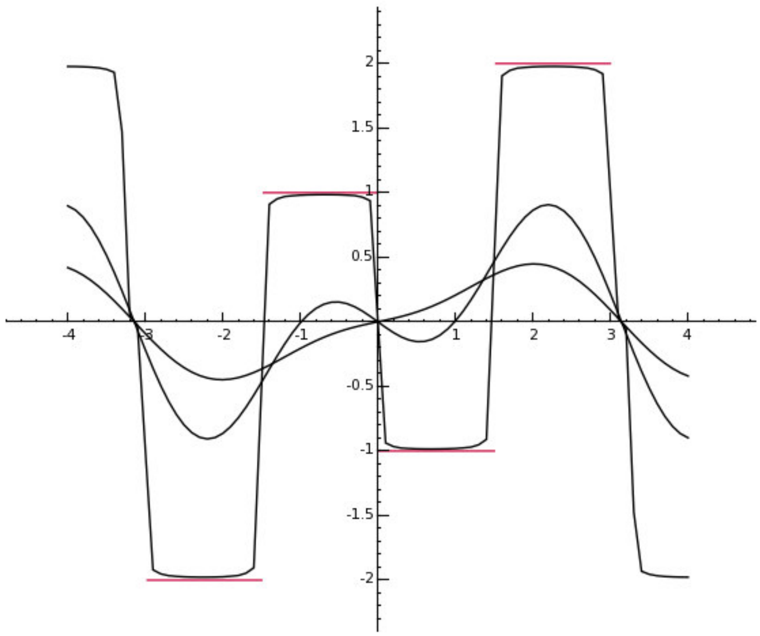

Figure 4: Heat Equation using Cesaro Filter

A modern application of Fourier Series:

Schrödinger's Equation for a Free Particle

$$\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial t^2}$$

Boundary Conditions : $\psi(0,t) = 0$ $\quad \psi(L,t) = 0$

Initial Conditions: $\psi(x,0) = f(x)$

$\hbar = 1.054\text{x}10^{-27}\, erg - s$

Schrödinger was one of the most brilliant physicists of the 20th century. His work on the time-dependent wave equation describes the relationship of space and time in quantum mechanical systems. It was based upon the assumption that a wave equation could be used to explain the behavior of atomic particles. Unlike the other equations we have looked at, like other physics principles, "the Schrödinger equation cannot be *derived*. It is simply a relation, like Newton's second law, that experience has shown to be true."[13] And to continue that analogy, the Schrödinger equation is to quantum mechanics what Newton's second law was to classical mechanics.

Solution to Schrödinger's Equation

We start off here by assuming the solution is separable.

$$\psi(x,t) = X(x)T(t)$$

$$\psi(x,t) = X(x)T(t)$$

$$\frac{\partial \psi}{\partial t} = XT'$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{i\hbar}{2m} X'' \qquad \text{Let } s = \frac{\hbar}{2m}$$

$$\frac{\partial \psi}{\partial t} = is\frac{\partial^2 \psi}{\partial x^2}$$

$$XT' = isX''T$$

By separation of variables we get :

$$\frac{T'}{isT} = \frac{X''}{X} = -c \qquad \text{Similar to the Heat and Wave Equations}$$

To solve for T : $\dfrac{T'}{T} = isc$

$$\ln(T) = -isct + d$$

$$T = e^{-isct+d} = e^{-isct}e^{d}$$

$$T = be^{-isct}$$

To solve for X : $\dfrac{X''}{X} = -c$

$$X'' = -cX \;\Rightarrow\; X'' + cX = 0$$

$$(D^2 + c)X = 0 \;\Rightarrow\; D = \pm\sqrt{c}i$$

$$X = a_1 \cos\sqrt{c}x + a_2 \sin\sqrt{c}x$$

Using the boundary conditions we get the same results as above :

$$X = a_2 \sin \sqrt{c}L \qquad \text{Where } a_1 = 0 \text{ and } c_n = \left(\frac{n\pi}{L}\right)^2 \quad n \in Z$$

Therefore we get $\psi(x,t) = X(x)T(t)$

$$\psi(x,t) = a_2 \sin\left(\sqrt{c}L\right) be^{-isct}$$

By the superposition principle

$$\psi(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) e^{-is\left(\frac{n\pi}{L}\right)^2 t}$$

Using the initial conditions :

$$\psi(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$$

$$\Rightarrow b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

Final Solution

$$\boxed{\begin{aligned} &\psi(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) e^{-is\left(\frac{n\pi}{L}\right)^2 t} \\ &\text{Where } \; b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx \end{aligned}}$$

Conclusion

After solving the heat equation, Fourier thought that all functions could be expressed using the Fourier series technique he had used. The key was using partial differential equations to represent actual phenomena (as X(x) and T(t) do above). Though partial differential equations had been in use prior to Fourier, "the techniques he used could not have arisen without his contribution: none of his contemporaries was as secure in their use of separation of variables nor as confident in infinite series solutions. The beauty of these techniques was - and remains - their wide applicability."[14] What

Fourier did with the heat equation allowed mathematicians to finally reconcile the two wave equation solutions provided by d'Alembert, Euler, and Bernoulli a century earlier. That was something they had not been able to accomplish because the solutions looked so radically different.[15]

Acknowledgements: I would like to thank Prof. W. David Joyner for his help with the SAGE representations of the heat and wave equations.

End Notes

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