Applications of Fourier Transforms in Solving Differential Equations

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History

- Daniel Bernoulli
  - Known for the Bernoulli Principle
  - Solved the wave equation in 1720's using techniques Fourier later would employ
  - Worked with Euler in St. Petersburg

History

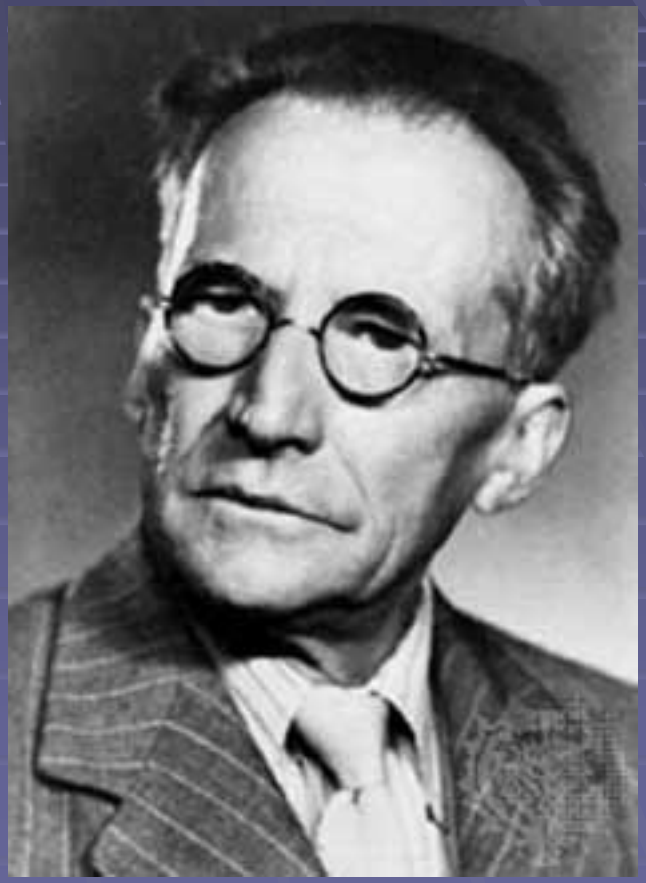
- Jean le Rond d'Alembert
  - French mathematician
  - Worked with Euler and Bernoulli on the wave equation
  - “Found” the Cauchy-Riemann equations decades before Cauchy or Riemann did

History

- Joseph Fourier
  - French mathematician
  - Served in Egypt while in Napoleon's Army
  - Solved the heat equation and won award for it in 1811, published work in 1822 in *Théorie analytique de la chaleur*

History

- Erwin Schrödinger
  - Vienna-born physicist
  - Artilleryman in WWI
  - Best known for “Schrödinger’s Cat” and the Schrödinger Equation
  - Won the Nobel Prize in 1933
  - Worked with Einstein
  - Fled Germany and Austria several times after “insulting” the Nazis



Concepts

- The following concepts were used

Fourier Coefficients

$$C_n = \frac{1}{P}\int_0^P f(x)e^{\frac{-2\pi i n x}{P}}\,dx$$

Fourier Series

$$FS(f)(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{2\pi i n x}{P}}$$

Superposition Principle

The Superposition Principle states that for a linear system, the sum of solutions for that system is also a solution.

Concepts

- Orthogonality of Sines

Given

$$f(x) = \sin\left(\frac{n\pi x}{L}\right); \; n \geq 1 \; ,$$

$$\frac{2}{L}\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = 0$$

for $m \neq n$

Concepts

- Proof of Orthogonality

$$\frac{2}{L}\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\left(\frac{\sin\left(\frac{(m-n)\pi x}{L}\right)}{\frac{2(m-n)\pi x}{L}} - \frac{\sin\left(\frac{(m+n)\pi x}{L}\right)}{\frac{2(m+n)\pi x}{L}}\right)\Bigg|_0^L$$

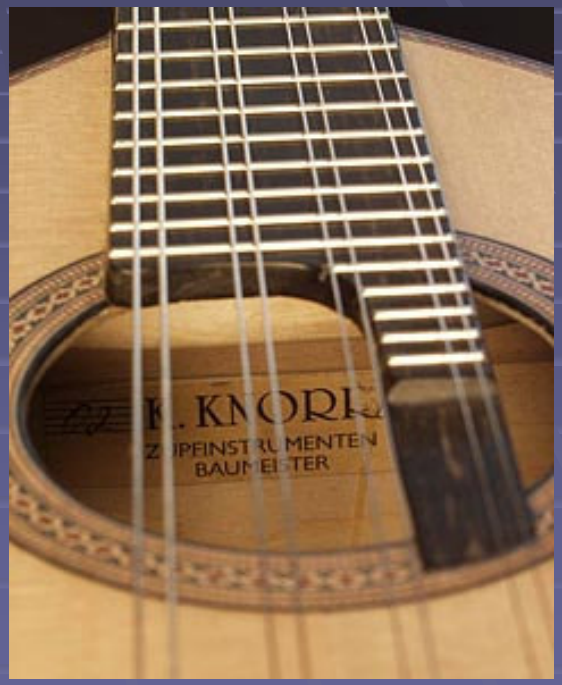
$$= \frac{2}{L}\left(\frac{\sin((m-n)\pi)}{2(m-n)\pi} - \frac{\sin((m+n)\pi)}{2(m+n)\pi}\right) - \frac{2}{L}(\sin(0) - \sin(0))$$

$$= \frac{2}{L}(0-0) - \frac{2}{L}(0-0)$$

$$= 0$$

Wave Equation

- People studied the wave equation because they were intrigued by string instruments.
- Bernoulli defined nodes and frequencies of oscillation



Derivation of Wave Equation

- To solve the wave equation, we assume that it is a separable equation:

$$u(x,t)=X(x)T(t)$$

The Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = \omega^2 \frac{\partial^2 u}{\partial x^2}$$

$$\text{Boundary conditions}: \quad u(0,t) = 0 \qquad u(L,t) = 0$$

$$\text{Initial Conditions}: \quad u(x,0) = f(x) \quad \frac{\partial u}{dt}(x,0) = g(x)$$

Derivation

$$\frac{\partial u}{\partial t} = XT' \qquad \frac{\partial^2 u}{\partial t^2} = XT''$$

$$\frac{\partial u}{\partial x} = X'T \quad \frac{\partial^2 u}{\partial x^2} = X''T$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = \omega^2 \frac{\partial^2 u}{\partial x^2} \Rightarrow T''X = \omega^2 X''T$$

By Separation of Variables we get :

$$\frac{T''}{\omega^2 T} = \frac{X''}{X} = -\lambda$$

because the only way the two sides will be equal is if they equal a constant

Solve for X as in the heat equation and you get

$$X = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right)$$

Solve for T

$$T'' = -\omega^2 \lambda T \;\Rightarrow\; T'' + \omega^2 \lambda T = 0$$

$$(D^2 + \omega^2 \lambda)T = 0 \;\Rightarrow\; D = \pm\left(\frac{\omega n \pi}{L}\right) i = \pm\gamma_n \;\; n \in \mathrm{Z}$$

$$\text{Therefore}: \; T = D_1 \cos(\gamma_n t) + D_2 \sin(\gamma_n t)$$

Thus we get

$$u(x,t) = X(x)T(t)$$

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right)\left(D_n \cos(\gamma_n t) + E_n \sin(\gamma_n t)\right) \quad \text{by Superposition Principle}$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left(D_n \cos(\gamma_n t) + E_n \sin(\gamma_n t)\right)$$

Using the Initial Conditions: $u(x,0) = f(x)$

$$u(x,0) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left(D_n \cos(\gamma_n 0) + E_n \sin(\gamma_n 0)\right) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$$

where $D_n = \frac{2}{L}\int_0^L f(x)\sin(\sqrt{\lambda_n}\,x)dx$

$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} \gamma_n \sin\left(\frac{n\pi}{L}x\right)\left(-D_n \sin(\gamma_n t) + E_n \cos(\gamma_n t)\right)$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \gamma_n \sin\left(\frac{n\pi}{L}x\right)\left(-D_n \sin(\gamma_n 0) + E_n \cos(\gamma_n 0)\right) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi}{L}x\right) = g(x)$$

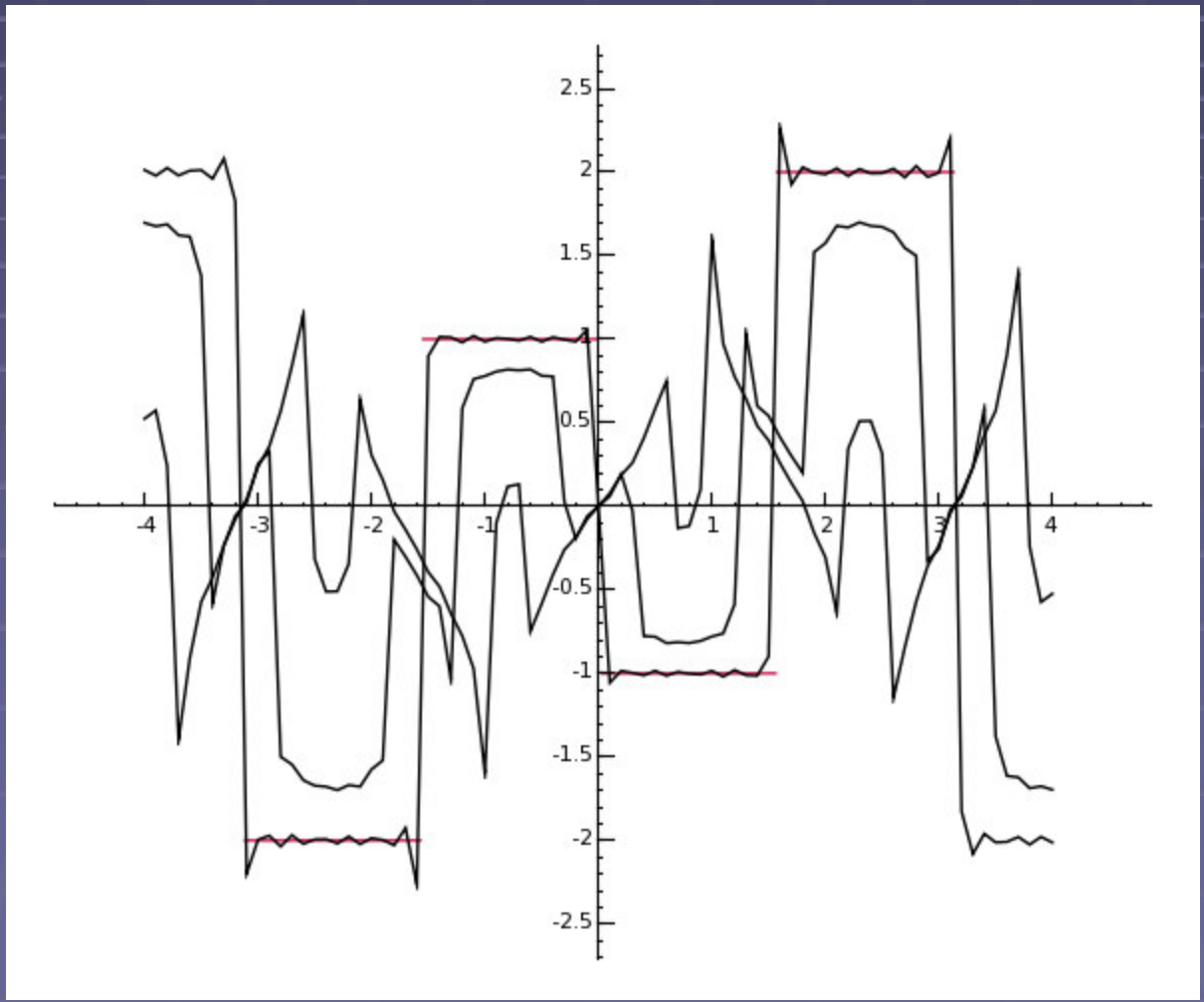
where $E_n = \frac{2}{\gamma_n L}\int_0^L g(x)\sin(\sqrt{\lambda_n}\,x)dx$

Final Solution

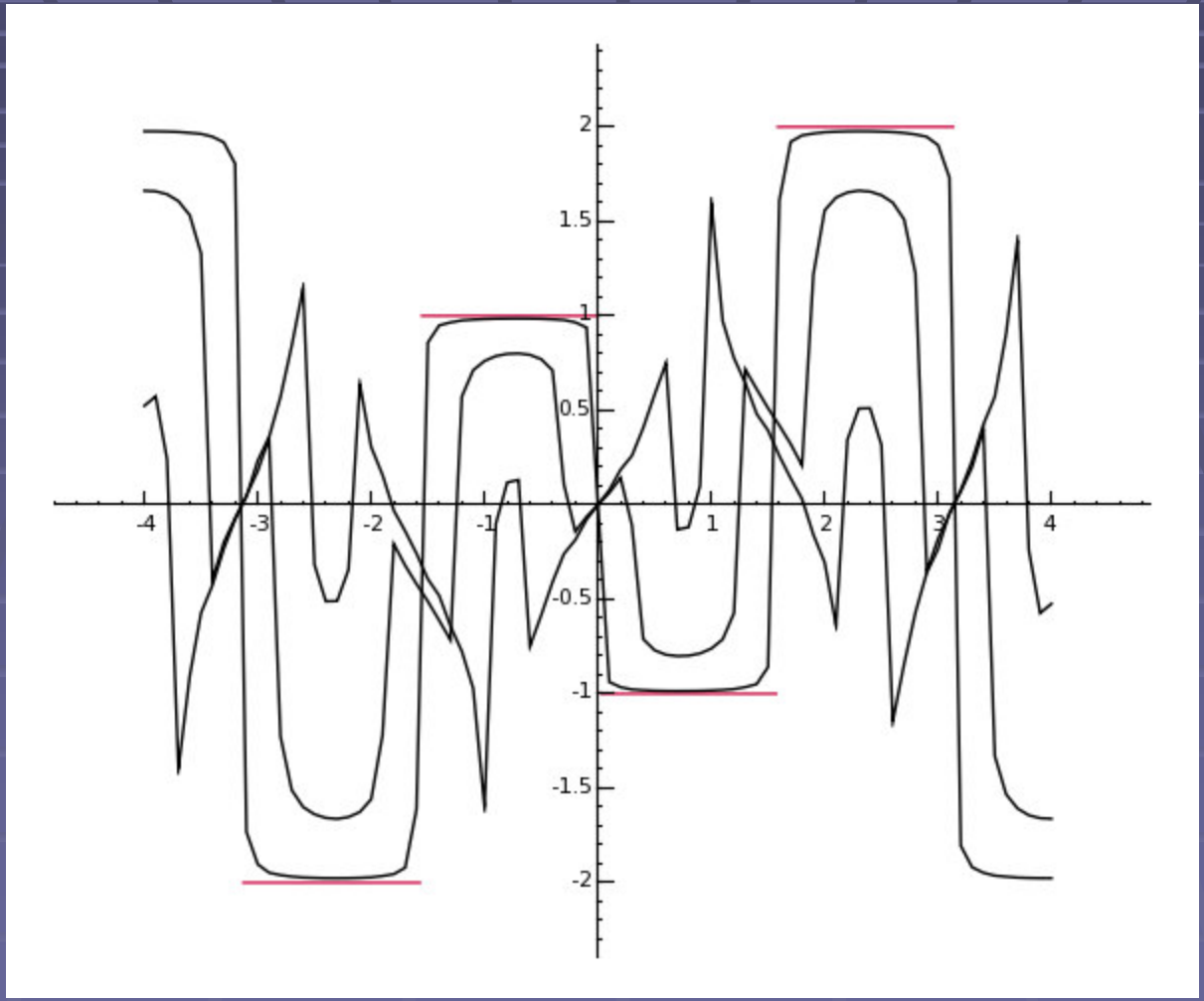
$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left(D_n \cos(\gamma_n t) + E_n \sin(\gamma_n t)\right)$$

where $D_n = \frac{2}{L}\int_0^L f(x)\sin(\sqrt{\lambda_n}\,x)dx$ and $E_n = \frac{2}{\gamma_n L}\int_0^L g(x)\sin(\sqrt{\lambda_n}\,x)dx$

Unfiltered Wave Equation



Filtered Wave Equation



Heat Equation

- The reason we solve the heat equation is to find out what happens to a length of metal/plastic when heat is applied (the temperature distribution over time)

Derivation of Heat Equation

- The assumption made by Bernoulli and adopted by Fourier was that the equation u(x,t) was separable:

$$u(x,t)=X(x)T(t)$$

The Heat Equation

$$\frac{\partial u}{\partial t} = \mathrm{k}\frac{\partial^2 u}{\partial \mathrm{x}^2}$$

$$\text{Boundary Conditions}: u(0,t)=0 \qquad u(L,t)=0$$

$$\text{Initial Conditions}: \quad u(x,0)=f(x)$$

Derivation

$$\frac{\partial u}{\partial t} = XT'$$

$$\frac{\partial u}{\partial x} = X'T \qquad \frac{\partial^2 u}{\partial x^2} = X''T$$

$$\frac{\partial \mathrm{u}}{\partial \mathrm{t}} = k\frac{\partial^2 u}{\partial x^2} \quad \Rightarrow \quad XT' = kX''T$$

The only way this can occur is if both sides equal a constant

$$\frac{\mathrm{T}'}{kT} = \frac{X''}{X} = -\lambda$$

Now Solve for X

$$X'' = -\lambda X \quad \Rightarrow \quad X'' + \lambda X = 0$$

$$(D^2 + \lambda)X = 0 \quad \Rightarrow \quad D = \pm\sqrt{\lambda}i$$

$$\Rightarrow \quad X = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$$

Using the boundary conditions we get :

$$X(0) = C_1 = 0 \quad \text{because } \cos(0) = 1 \text{ and } \sin(0) = 0$$

$$X(L) = C_2 \sin\sqrt{\lambda}L = 0 \quad \Rightarrow \quad \sin\sqrt{\lambda}L = 0 \text{ when } \sqrt{\lambda}L = n\pi \Rightarrow \lambda_n = \left(\frac{\mathrm{n}\pi}{\mathrm{L}}\right)^2 \quad \mathrm{n} \in \mathrm{Z}$$

$$\text{Therefore: } X = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) \quad \text{by the Superposition Principle}$$

Now Solve for T

$$T' = -\lambda kT \quad \Rightarrow \quad T' + \lambda kT = 0$$

$$(D + \lambda k)T = 0 \quad \Rightarrow \quad D = -\lambda k$$

$$\Rightarrow \quad T = Be^{-\lambda kt}$$

Thus, we get : $u(x,t) = X(x)T(t)$

$$u(x,t) = Be^{-\lambda kt} \sum_{n=1}^{\infty} A_n \sin(\sqrt{\lambda_n}x) = \sum_{n=1}^{\infty} A_n \sin\left(\sqrt{\lambda_n}x\right)e^{-\lambda_n kt}$$

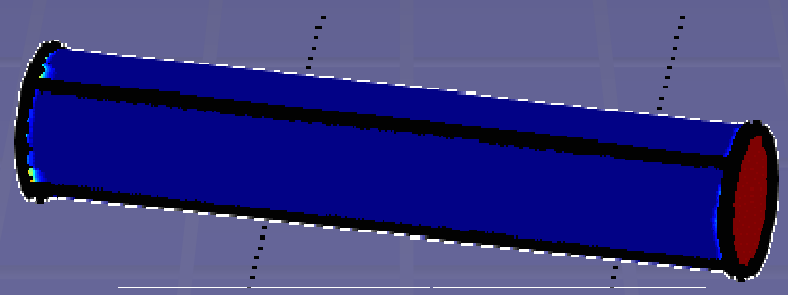
Using the initial condition : $u(x,0) = f(x)$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(\sqrt{\lambda_n}\, x) = f(x)$$

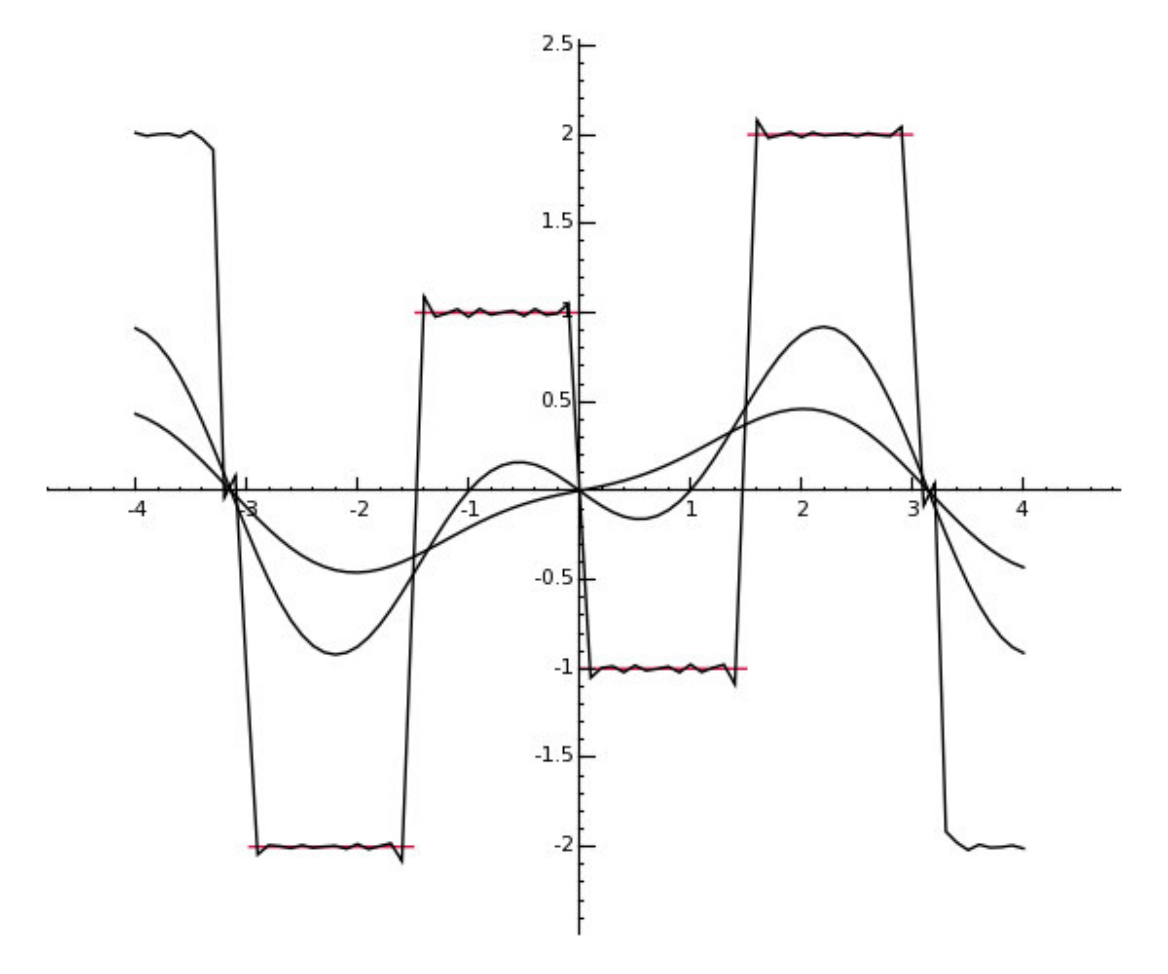
$$\Rightarrow \quad A_n = \frac{1}{L}\int_0^L f(x)\sin(\sqrt{\lambda_n}\, x)dx$$

Final Solution

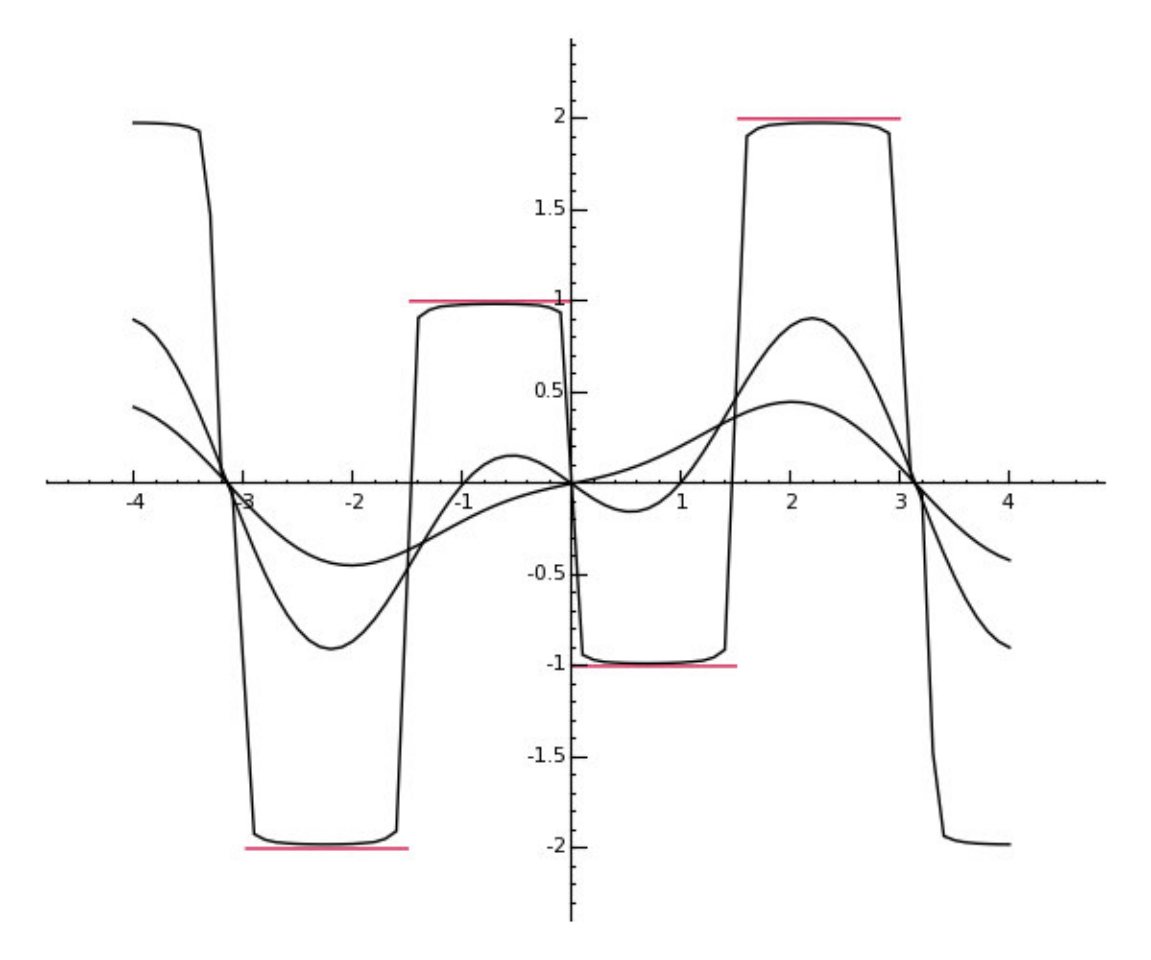
$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(\sqrt{\lambda_n}\, x)e^{-\lambda_n kt} \quad \text{where} \quad A_n = \frac{1}{L}\int_0^L f(x)\sin(\sqrt{\lambda_n}\, x)dx$$



Unfiltered Heat Equation



Filtered Heat Equation



Euler and d'Alembert

- They too solved the wave equation, but by using "arbitrary functions representing two waves, one moving along the string to the right, the other to the left, with a velocity equal to the constant c."
- Bernoulli's solution was a summation of sine waves
- No one could tell if the two solutions could be reconciled

Until Fourier Came Along

- "Fourier showed that almost any function, when regarded as a periodic function over a given interval, can be represented by a trigonometric series of the form

$$f(x)=a_o+a_1\cos x+a_2\cos 2x+a_3\cos 3x\ldots + b_1\sin x+b_2\sin 2x+b_3\sin 3x\ldots$$

where the coefficients a_i and b_i can be found from f(x) by computing certain integrals."

Modern Application:
Schrödinger

- The Schrödinger Equation was published in a series of papers on wave mechanics in 1926
- The solutions to the Schrödinger Equation are wave functions that express the probability that a particle (specifically the electron of a hydrogen atom) will be in a certain location at any observation

Solution to Schrödinger

- As with the heat and wave equations, we assume that the Schrödinger Equation is separable:

$$u(x,t)=X(x)T(t)$$

The Schrödinger Equation

$$\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial t^2}$$

$$\text{Boundary Conditions}: \quad \psi(0,t)=0 \qquad \psi(L,t)=0$$

$$\text{Initial Conditions}: \quad \psi(x,0)=f(x)$$

$$\hbar = 1.054\text{x}10^{-27}\, erg-s$$

Solution

Assume Ψ(x,t) is separable

$$\psi(x,t) = X(x)T(t)$$

$$\frac{\partial \psi}{\partial t} = XT'$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{i\hbar}{2m} X''$$

Let $s = \frac{\hbar}{2m}$

$$\frac{\partial \psi}{\partial t} = is \frac{\partial^2 \psi}{\partial x^2}$$

$$XT' = isX''T$$

By separation of variables we get :

$$\frac{T'}{isT} = \frac{X''}{X} = -c$$

Similar to the Heat and Wave Equations

Solve for T:

$$\text{To solve for T}: \quad \frac{T'}{T} = isc$$

$$\ln(T) = -isct + d$$

$$T = e^{-isct+d} = e^{-isct} e^{d}$$

$$T = be^{-isct}$$

Solve for X:

$$\text{To solve for X}: \quad \frac{X''}{X} = -c$$

$$X'' = -cX \;\Rightarrow\; X'' + cX = 0$$

$$(D^2 + c)X = 0 \;\Rightarrow\; D = \pm\sqrt{c}i$$

$$X = a_1 \cos(\sqrt{c}x) + a_2 \sin(\sqrt{c}x)$$

Using the boundary conditions we get the same results as before:

$$X = a_2 \sin\sqrt{c}L \qquad \text{Where } a_1 = 0 \quad \text{and } c_n = \left(\frac{n\pi}{L}\right)^2 \quad n \in Z$$

Therefore we get $\psi(x,t) = X(x)T(t)$

$$\psi(x,t) = a_2 \sin\left(\sqrt{c}L\right) b e^{-isct}$$

By the superposition principle

$$\psi(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) e^{-is\left(\frac{n\pi}{L}\right)^2 t}$$

Using the initial conditions:

$$\psi(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$$

$$\Rightarrow b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

Final Solution

$$\psi(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) e^{-is\left(\frac{n\pi}{L}\right)^2 t} \quad \text{Where} \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

SAGE

- The graphs were produced using the open source program SAGE by Prof. W. David Joyner, USNA

http://modular.math.washington.edu/sage

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