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# Understanding Parseval's Identity

Final Report for Capstone Project

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Professor Joyner on
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The goal of this paper is to state, prove, and
show applications for Parseval's Identity

# PARSEVAL'S IDENTITY

## 1. INTRODUCTION

I am writing this paper to learn and understand more about Fourier series and Fourier analysis. My focus will be on Parseval's Identity.

## 1.1 THESIS

This paper will introduce, state, and prove Parseval's Identity and show how it relates to applications.

## 1.2. BACKGROUND ON MARC-ANTOINE PARSEVAL

Born into a family of high standing in France, Parseval grew up in a family of wealthy land-owners. Parseval, who considered himself a squire in his younger years, lived from 1755-1836. Not much was known about Parseval's personal life. It is known that he once fled France from Napoleon when he published poetry about Napoleon's regime and Napoleon ordered his arrest. Little is known about Parseval's personal life. There is no record of his own family in his older years or other personal information such as hobbies and interests. Parseval's mathematic contributions consisted of only five publications which were presented to the French Academy of Science. The second of these publications is where he proved the result known today as Parseval's Identity or Parseval's Theorem.[1]

Parseval's theorem occurs in many different contexts. In many problems, Parseval's theorem breaks down a given function into linear combinations of orthogonal eigenfunctions. Parseval's theorem says that the total energy of the original function is the sum of the different eigenfunctions' energies. Although Parseval's theorem may be applied to general orthogonal basis of eigenfunctions, it is mainly discussed with Fourier series and Fourier transform.[2]

## 1.3 UNDERSTANDING PARSEVAL'S THEOREM

Real and Imaginary Numbers

Let $a \in \square$ on the x-axis and $b \in \square$ on the y-axis.

Let $z = a + bi$. Let the complex conjugate of $z$ be $\bar{z}$.

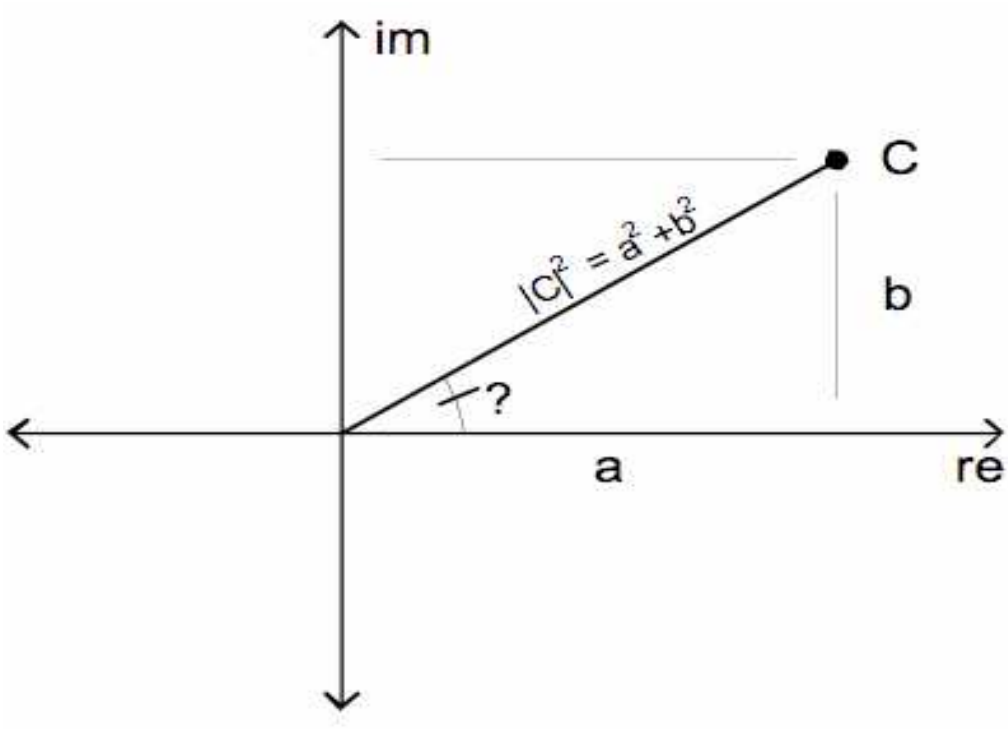
So $\bar{z} = a - bi$

The magnitude of $z$ is equal to the square root of $a^2 + b^2$

This is written as $|z| = \sqrt{a^2 + b^2}$. We have:

$$
\begin{aligned}
z\bar{z} &= (a+bi)(a-bi) \\
&= a^2 + abi - abi + b^2 \\
&= a^2 + b^2 \\
&= |z|^2
\end{aligned}
$$

c=z for the graph below.



Inner Product on $L^2(I)$

Suppose $f, g$ are two square integrable functions on the interval I=[a,b].

ie. $\int_a^b |f(x)|^2 \, dx$ exists and is finite. (We say that such a function belongs to $L^2(I)$.)

Their inner product is defined by

$$(f, g) = \int_a^b f(x)\overline{g(x)}dx$$

Properties of the Inner Product

1. $(c_1 f_1 + c_2 f_2, g) = c_1(f_1, g) + c_2(f_2, g)$ where $c_1, c_2 \in \mathbb{C}$
2. $(f, f) \geq 0$
3. $(f, g) = \overline{(g, f)}$

Follows that for $\alpha \in \mathbb{C}$

$$(f, \alpha g) = \int_a^b f(x)\overline{\alpha g(x)}dx$$

$$(f, \alpha g) = \int_a^b f(x)\overline{\alpha}\,\overline{g(x)}dx$$

$$(f, \alpha g) = \overline{\alpha}\int_a^b f(x)\overline{g(x)}dx$$

$$(f, \alpha g) = \overline{\alpha}(f, g)$$

Using the inner product, we define

$\|f\| = (f, f)^{1/2}$ to be the "magnitude" or norm of a function.

(This is most commonly denoted by $\|f\|_2$ )

Then $\|f - g\|$ is defined to be the distance between $f$ and $g$.

Orthonormal Family of Functions

Definition: A family of functions $\{\varphi_k : k = 0, 1, 2...\}$ is orthonormal if the following properties are satisfied

(1) $(\varphi_k, \varphi_l) = 0$ if $k \neq l$

(2) $(\varphi_k, \varphi_k) = 1$ (i.e. $\|\varphi_k\| = 1$) $\forall$ k

Suppose $\{\varphi_0, \varphi_{1,...}\varphi_n\}$ is an orthonormal set of functions and $f$ is a linear combination of the $\varphi_m$. Then it is easy to compute the coefficients $c_m$.

Suppose $f = \sum_{k=0}^{n} c_k \varphi_k$

$$= c_0\varphi_0 + c_1\varphi_1 + ... + c_n\varphi_n$$

To find $c_0$ : take the inner product of both sides with $\varphi_0$

$$(f, \varphi_0) = (c_0\varphi_0 + c_1\varphi_1 + ... + c_n\varphi_n, \varphi_0)$$

$$= c_0(\varphi_0, \varphi_0) + c_1(\varphi_1, \varphi_0) + ... + c_n(\varphi_n, \varphi_0)$$

Remember from above that $(\varphi_k, \varphi_l) = 0$ if $k \neq l$ and

$$(\varphi_k, \varphi_k) = 1 \text{ (i.e. } \|\varphi_k\| = 1)$$

So every inner product is 0 except for $(\varphi_0, \varphi_0)$ which is equal to 1.

$(f, \varphi_0) = (c_0 + 0 + ... + 0)$

Therefore $c_0 = (f, \varphi_0)$

In general we get $c_m = (f, \varphi_m)$

Definition: The numbers $c_m$ are called the Fourier coefficients of $f$ with respect to the orthonormal set $\{\varphi_0, \varphi_1, ..., \varphi_n\}$

Next suppose $\{\varphi_0, \varphi_1, ..., \varphi_n\}$ is an orthonormal set and $f$ is a function in $L^2(I)$.
The following theorem is the crucial step in proving Parseval's theorem. It says that among all linear combinations $b_0\varphi_0 + b_1\varphi_1 + ... + b_n\varphi_n$ the one that best approximates $f$ is obtained by letting $b_i$ be the Fourier coefficients of $f$ with respect to $\{\varphi_0, \varphi_1, ..., \varphi_n\}$.

<u>Theorem</u>

Suppose $f$ belongs to $L^2(I)$.

i.e. if $s_n = c_0\varphi_0 + c_1\varphi_1 + ... + c_n\varphi_n$ where $c_m = (f, \varphi_m)$

and $t_n = b_0\varphi_0 + b_1\varphi_1 + ... + b_n\varphi_n$ where $b_m \in \square$

then $\|f - s_n\| \leq \|f - t_n\|$

Note: $s_n$ is the closest approximation to $f$ than any other generic or random linear combination of the $\varphi_n$.

Note: $t_n$ is a generic or random combination of the $\varphi_n$.

PROOF

$\|f - t_n\|^2 = (f - t_n, f - t_n)$

Use the definition of the inner product to solve the above equation.

$$= \int_a^b (f(x) - t_n(x))(\overline{f(x)} - \overline{t_n(x)})dx$$

$$= \int_a^b (f(x)(\overline{f(x)} - \overline{t_n(x)}) - t_n(x)(\overline{f(x)} - \overline{t_n(x)})dx$$

$$= \int_a^b (f(x)(\overline{f(x)} - \overline{t_n(x)})dx - \int_a^b t_n(x)(\overline{f(x)} - \overline{t_n(x)})dx$$

$$= (f, f - t_n) - (t_n, f - t_n)$$

$$= (f, f) - (f, t_n) - (t_n, f) + (t_n, t_n)$$

Each term is now individually expanded.

- $(t_n, t_n) = (b_0\varphi_0 + b_1\varphi_1 + ... + b_n\varphi_n, b_0\varphi_0 + b_1\varphi_1 + ... + b_n\varphi_n)$
  $= (b_0\varphi_0, b_0\varphi_0) + (b_1\varphi_1, b_1\varphi_1) + ... + (b_n\varphi_n, b_n\varphi_n)$

There should be $n^2$ terms, however there are only n terms since all other terms cancel out where $(b_n\varphi_n, b_m\varphi_m)$ and $n \neq m$.

So $(t_n, t_n) = b_0\overline{b_0}(\varphi_0, \varphi_0) + b_1\overline{b_1}(\varphi_1, \varphi_1) + ... + b_n\overline{b_n}(\varphi_n, \varphi_n)$

$$= b_0\overline{b_0} + b_1\overline{b_1} + ... + b_n\overline{b_n}$$

$$= |b_0|^2 + |b_1|^2 + ... + |b_n|^2 = \sum_{k=0}^{n} |b_k|^2$$

- $(f, t_n) = (f, b_0\varphi_0 + b_1\varphi_1 + ... + b_n\varphi_n)$
  $= (f, b_0\varphi_0) + (f, b_1\varphi_1) + ... + (f, b_n\varphi_n)$
  $= \overline{b_0}(f, \varphi_0) + \overline{b_1}(f, \varphi_1) + ... + \overline{b_n}(f, \varphi_n)$

Remember that $(f, \varphi_m) = c_m$.

So $(f, t_n) = \overline{b_0}c_0 + \overline{b_1}c_1 + ... + \overline{b_n}c_n$

$$= \sum_{k=0}^{n} \overline{b_k}c_k$$

- $(t_n, f) = \overline{(f, t_n)}$

$$= \overline{\sum_{k=0}^{n} \overline{b_k} c_k}$$

$$= \sum_{k=0}^{n} \overline{c_k} b_k$$

So $\|f - t_n\|^2 = \|f\|^2 - (f, t_n) - (t_n, f) + (t_n, t_n)$

$$\|f - t_n\|^2 = \|f\|^2 - \sum_{k=0}^{n} \overline{b_k} c_k - \sum_{k=0}^{n} \overline{c_k} b_k + \sum_{k=0}^{n} |b_k|^2$$

Now we complete the square by adding and subtracting $\sum_{k=0}^{n} |c_k|^2$

$$\|f - t_n\|^2 = \|f\|^2 - \sum_{k=0}^{n} |c_k|^2 + \underbrace{\sum_{k=0}^{n} |c_k|^2 - \sum_{k=0}^{n} \overline{b_k} c_k - \sum_{k=0}^{n} \overline{c_k} b_k + \sum_{k=0}^{n} |b_k|^2}_{\text{"completing the square"}} \quad (2)$$

We next show that

$$\sum_{k=0}^{n} |b_k - c_k|^2 = \sum_{k=0}^{n} |c_k|^2 - \sum_{k=0}^{n} \overline{b_k} c_k - \sum_{k=0}^{n} \overline{c_k} b_k + \sum_{k=0}^{n} |b_k|^2$$

$\sum_{k=0}^{n} |b_k - c_k|^2$ can be factored as follows

$$\sum_{k=0}^{n} |b_k - c_k|^2 = \sum_{k=0}^{n} (b_k - c_k)(\overline{b_k} - \overline{c_k})$$

$$\sum_{k=0}^{n} |b_k - c_k|^2 = \sum_{k=0}^{n} (b_k \overline{b_k} - c_k \overline{b_k} - b_k \overline{c_k} + c_k \overline{c_k})$$

Remember $|b_k|^2 = b_k \overline{b_k}$.

$$\sum_{k=0}^{n} |b_k - c_k|^2 = \sum_{k=0}^{n} (|b_k|^2 - c_k \overline{b_k} - b_k \overline{c_k} + |c_k|^2)$$

$$\sum_{k=0}^{n} |b_k - c_k|^2 = \sum_{k=0}^{n} |b_k|^2 - \sum_{k=0}^{n} c_k \overline{b_k} - \sum_{k=0}^{n} b_k \overline{c_k} + \sum_{\mathrm{k}=0}^{\mathrm{n}} |c_k|^2$$

So we can reduce the equation (2) to

$$\|f - t_n\|^2 = \|f\|^2 - \sum_{k=0}^{n} |c_k|^2 + \sum_{k=0}^{n} |b_k - c_k|^2$$

For which choice of $b_k$ does this become a minimum?

The minimum is attained where all the numbers $|b_k - c_k| = 0$

*i.e.* when $b_k = c_k$

Theorem: Let $\{\varphi_0, \varphi_1, \varphi_2, ...\}$ be an orthonormal set in $\mathrm{L}^2(I)$ where $I = [a,b]$ and $\mathrm{L}^2(I) = \{f : I \to \square$ such that $\int_a^b |f(x)|^2 dx < \infty\}$. Recall the definitions for the inner product and the norm:

$$(f,g) = \int_a^b f(x)\overline{g(x)}dx, \ \|f\| = (f,f)^{\frac{1}{2}} = [\int_a^b |f(x)|^2 dx]^{\frac{1}{2}}$$

Suppose $f \in \mathrm{L}^2(I)$ and let $c_k = (f, \varphi_k)$

(1)$\sum_{k=0}^{\infty} |c_k|^2 \le \|f\|^2$ (Bessel's Inequality)

(2)Assume that furthermore $\|f - s_n\| \underset{n\to\infty}{\longrightarrow} 0$

where $s_n(x) = \sum_{k=0}^{n} c_k \varphi_k(x)$

then $\sum_{k=0}^{\infty} |c_k|^2 = \|f\|^2$. (Parseval's Identity)

PROOF OF (1):

From previous proof, if $t_n = \sum_{k=0}^{n} b_k \varphi_k(x)$

$$\|f - t_n\| = \|f\|^2 - \sum_{k=0}^{\infty} |c_k|^2 + \sum_{k=0}^{n} |b_k - c_k|^2$$

So by replacing $t_n$ by $s_n$ (which means $b_k = c_k$)

we get $\|f - s_n\|^2 = \|f\|^2 - \sum_{k=0}^{n} |c_k|^2$

which can be rearranged into

$$\sum_{k=0}^{n} |c_k|^2 = \|f\|^2 - \|f - s_n\|^2 \le \|f\|^2$$

Since this is true for $\forall n$, we get $\lim_{n\to\infty} \sum_{k=0}^{n} |c_k|^2 \le \|f\|^2$, *i.e.*, $\sum_{k=0}^{\infty} |c_k|^2 \le \|f\|^2$ (Bessel's Inequality)

PROOF OF (2):

Now assume that $\|f - s_n\| \underset{n\to\infty}{\longrightarrow} 0$. Then we get

$$\sum_{k=0}^{n} |c_k|^2 = \|f\|^2 - \|f - s_n\|^2 \underset{n\to\infty}{\Longrightarrow} \sum_{k=0}^{n} |c_k|^2 = \|f\|^2 \text{ (Parseval's Identity)}$$

We present below an example as well as some special cases of Parseval's Identity. First we present an application of Parseval's Identity. Consider the set of functions $\{\varphi_0, \varphi_1, \varphi_2, ...\}$ in $L^2([0, 2\pi])$ given by

$$\varphi_0(x) = \frac{1}{\sqrt{2\pi}} \qquad \varphi_{2n-1}(x) = \frac{\cos(nx)}{\sqrt{\pi}} \qquad \varphi_{2n}(x) = \frac{\sin(nx)}{\sqrt{\pi}}.$$

We show below that this set is orthonormal.

$$\begin{aligned}\int_0^{2\pi} \varphi_0(x)\varphi_0(x) &= \int_0^{2\pi} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} dx \\ &= \int_0^{2\pi} \frac{1}{\sqrt{4\pi^2}} = \frac{x}{2\pi} |_0^{2\pi} \\ &= \frac{2\pi}{2\pi} - \frac{0}{2\pi} = 1\end{aligned}$$

$$\begin{aligned}\int_0^{2\pi} \varphi_{2n-1}(x)\varphi_{2n-1}(x)dx &= \int_0^{2\pi} \frac{\cos(nx)}{\sqrt{\pi}} \frac{\cos(nx)}{\sqrt{\pi}} dx \\ &= \frac{\sin(2n\pi)\cos(2n\pi) + 2n\pi}{2n\pi} |_0^{2\pi} \\ &= 0 + \frac{2n\pi}{2n\pi} = 1\end{aligned}$$

Similarly, one shows the following:

$$\int_0^{2\pi} \varphi_0(x)\varphi_{2n-1}(x)dx = \int_0^{2\pi} \frac{1}{\sqrt{2\pi}} \frac{\cos(nx)}{\sqrt{\pi}} dx = 0$$

$$\int_0^{2\pi} \varphi_0(x)\varphi_{2n}(x)dx \int_0^{2\pi} \frac{1}{\sqrt{2\pi}} \frac{\sin(nx)}{\sqrt{\pi}} dx = 0$$

$$\int_0^{2\pi} \varphi_{2n}(x)\varphi_{2n}(x)dx = \int_0^{2\pi} \frac{\sin(nx)}{\sqrt{\pi}} \frac{\sin(nx)}{\sqrt{\pi}} dx = 1$$

$$\int_0^{2\pi} \varphi_{2n-1}(x)\varphi_{2m-1}(x)dx = \int_0^{2\pi} \frac{\cos(nx)}{\sqrt{\pi}} \frac{\cos(mx)}{\sqrt{\pi}} dx = 0 \text{ where m} \neq \text{n}$$

$$\int_0^{2\pi} \varphi_{2n}(x)\varphi_{2m}(x)dx = \int_0^{2\pi} \frac{\sin(nx)}{\sqrt{\pi}} \frac{\sin(mx)}{\sqrt{\pi}} dx = 0 \text{ where m} \neq \text{n}$$

$$\int_0^{2\pi} \varphi_{2n}(x)\varphi_{2m-1}(x)dx = \int_0^{2\pi} \frac{\sin(nx)}{\sqrt{\pi}} \frac{\cos(mx)}{\sqrt{\pi}} dx = 0 \text{ where m} \neq \text{n}$$

Theorem: Suppose $f \in \mathrm{L}^2([0, 2\pi])$

and let $s_n(x) = \sum_{k=0}^{n} c_k \varphi_k(x)$ where $c_k = (f, \varphi_k)$.

Then $\|f - s_n\| \underset{n\to\infty}{\Longrightarrow} 0$.

The proof is omitted. See W. Rudin, Theorem 8.16.

Let $f(x) = x$ on $[0,2\pi]$

Compute $c_k$

$$c_0 = (f, \varphi_0) = \int_0^{2\pi} f(x)\varphi_0(x)dx = \int_0^{2\pi} x \frac{1}{\sqrt{2\pi}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{x^2}{2} \Big|_0^{2\pi} = \frac{4\pi^2}{2\sqrt{2\pi}} = \frac{2\pi^2}{\sqrt{2\pi}}$$

$$c_{2n-1} = (f, \varphi_{2n-1}) = \int_0^{2\pi} x \frac{\cos(nx)}{\sqrt{\pi}} dx$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{2\pi} x\cos(nx)dx$$

Now we are going to use integration by parts to compute the above integral.

$u = x \qquad dv = \cos(nx)dx$

$du = dx \qquad v = \frac{1}{n}\sin(nx)$

$$c_{2n-1} = (f, \varphi_{2n-1}) = \frac{1}{\sqrt{\pi}} [\frac{1}{n} x\sin(nx) \Big|_0^{2\pi} - \int_0^{2\pi} \frac{1}{n}\sin(nx)dx]$$

$$= \frac{1}{\sqrt{\pi}} \frac{1}{n^2} \cos(nx) \Big|_0^{2\pi} = 0$$

$$c_{2n} = (f, \varphi_{2n}) = \int_0^{2\pi} x \frac{\sin(nx)}{\sqrt{\pi}} dx$$
$$= \frac{1}{\sqrt{\pi}} \int_0^{2\pi} x \sin(nx) dx$$

Once again we use integration by parts to compute the above integral.

$u = x \qquad dv = \sin(nx)dx$

$du = dx \qquad v = \frac{1}{n}\cos(nx)$

$$c_{2n} = (f, \varphi_{2n}) = \frac{1}{\sqrt{\pi}}[-\frac{1}{n} x\cos(nx)\,|_0^{2\pi} + \int_0^{2\pi} \frac{1}{n}\cos(nx)dx]$$
$$= \frac{1}{\sqrt{\pi}}(-\frac{1}{n}2\pi + \frac{1}{n^2}\sin(nx)\,|_0^{2\pi} = -\frac{2\pi}{\sqrt{\pi}}\frac{1}{n}$$

$$f(x) = x \sim \underbrace{\frac{2\pi^2}{\sqrt{2\pi}}}_{c_0}\frac{1}{\sqrt{2\pi}} - \sum_{n=1}^{\infty} \underbrace{\frac{2\pi}{\sqrt{\pi}}\frac{1}{n}}_{c_{2n}}\frac{\sin(nx)}{\sqrt{\pi}}$$

Combining the above two theorems, we obtain Parseval's Identity:

$\sum_{k=0}^{\infty} |c_k|^2 = \|f\|^2$. Since

$\sum_{k=0}^{\infty} |c_k|^2 = \frac{4\pi^4}{2\pi} + 4\pi\sum_{n=1}^{\infty}\frac{1}{n^2} = 2\pi^3 + 4\pi\sum_{n=1}^{\infty}\frac{1}{n^2}$ and

$\|f\|^2 = \int_0^{2\pi} f(x)\overline{f(x)}dx = \int_0^{2\pi} x^2 dx = \frac{x^3}{3}\,|_0^{2\pi} = \frac{8\pi^3}{3}$ we get

$$2\pi^3 + 4\pi\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{8\pi^3}{3}$$

Rearrange with algebra to get

$$4\pi\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2\pi^3}{3} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Generally consider the space $L^2[-T,T]$

The functions

$$\varphi_0(x)=\frac{1}{\sqrt{2T}},\quad \varphi_{2n-1}(x)=\frac{1}{\sqrt{T}}\cos(\frac{n\pi x}{T}),\quad \varphi_{2n}(x)=\frac{1}{\sqrt{T}}\sin(\frac{n\pi x}{T})$$

form a complete orthonormal set. (Complete means the hypothesis to the previous theorem is satisfied.)

Let $f(x)\sim\frac{a_0}{2}+\sum a_n\cos(\frac{n\pi x}{T})+b_n\sin(\frac{n\pi x}{T})$ be the Fourier series of $f$

where $a_0=\frac{1}{T}\int_{-T}^{T}f(x)dx$

$$a_n=\frac{1}{T}\int_{-T}^{T}f(x)\cos(\frac{n\pi x}{T})dx$$

$$b_n=\frac{1}{T}\int_{-T}^{T}f(x)\sin(\frac{n\pi x}{T})dx.$$

In general, Parseval's Identity says that

$$\int_{-T}^{T}|f(x)|^2dx=\sum_{n=0}^{\infty}|c_n|^2 \quad *$$

where $c_n=(f,\varphi_n)$. We compute:

$$\begin{aligned}c_0=(f,\varphi_0)&=\int_{-T}^{T}f(x)\varphi_0(x)dx\\&=\frac{1}{\sqrt{2T}}\int_{-T}^{T}f(x)dx=\frac{T}{\sqrt{2T}}a_0\\&=\frac{1}{\sqrt{2T}}\frac{T}{T}\int_{-T}^{T}f(x)dx=\frac{T}{\sqrt{2T}}\frac{1}{T}\int_{-T}^{T}f(x)dx=\frac{T}{\sqrt{2T}}a_0\end{aligned}$$

$$\begin{aligned}c_{2n-1}&=(f,\varphi_{2n-1})\\&=\int_{-T}^{T}f(x)\frac{1}{\sqrt{T}}\cos(\frac{n\pi x}{T})dx\\&=\frac{1}{\sqrt{T}}\int_{-T}^{T}f(x)\cos(\frac{n\pi x}{T})dx=\sqrt{T}a_n\end{aligned}$$

Similarly

$$c_{2n}=(f,\varphi_{2n})=\sqrt{T}b_n$$

So $\sum_{n=0}^{\infty}|c_n|^2 = (\frac{T}{\sqrt{2T}}|a_0|)^2 + \sum_{n=1}^{\infty}(\sqrt{T}\,|a_n|)^2 + (\sqrt{T}\,|b_n|)^2$

$$= \frac{T}{2}|a_0|^2 + \sum_{n=1}^{\infty} T(|a_n|^2 + |b_n|^2)$$

$$= T(\frac{1}{2}|a_0|^2 + \sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2)$$

So * gives

$$\frac{1}{\mathrm{T}}\int_{-T}^{T}|f(x)|^2\,dx = \frac{1}{2}|a_0|^2 + \sum_{n=1}^{\infty}|a_n|^2 + |b_n|^2$$

This is Parseval's Identity for real Fourier series.

<u>Parseval's Identity for complex fourier series</u>

Consider the set of functions

$$\varphi_{\mathrm{n}}(x) = \frac{1}{\sqrt{2T}}e^{\frac{in\pi x}{T}} \quad n \in \square .$$

This is a complete orthonormal set in $L^2([-T,T])$.

If $f(x) = \sum_{n=-\infty}^{\infty} c_n\varphi_n(x)$ is the Fourier series expansion of $f(x)$ with respect to this set, Parseval's Identity gives that

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\varphi_n(x),\ \int_{-T}^{T}|f(x)|^2\,dx = \sum_{n=-\infty}^{\infty}|c_n|^2 \quad *$$

Recall that the complex Fourier series of $f(x)$ over $[-T,T]$ is given by

$$f(x) \sim \sum_{n=-\infty}^{\infty} d_n e^{\frac{in\pi x}{T}}$$

where $d_n = \frac{1}{2T}\int_{-T}^{T} f(x)e^{\frac{-in\pi x}{T}}dx$

$$= \frac{1}{2T}\sqrt{2T}\int_{-T}^{T} f(x)\underbrace{\frac{1}{\sqrt{2T}}e^{\frac{-in\pi x}{T}}}_{\varphi_n(x)}dx$$

$$= \frac{1}{\sqrt{2T}}(f,\varphi_n) = \frac{1}{\sqrt{2T}}c_n$$

So * gives $\int_{-T}^{T} |f(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} |c_n|^2$

$$= 2T \sum_{n=-\infty}^{\infty} |d_n|^2$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} |d_n|^2 = \frac{1}{2T} \int_{-T}^{T} |f(x)|^2 \, dx$$

This is Parseval's Identity for complex Fourier Series.

Finally we present Parseval's Identity for the discrete Fourier Transfrom.

Discrete Fourier Transform

$h = (h_0, h_1, ..., h_{n-1})$ n-tuple of complex numbers

Then their Discrete Fourier Transform is the n-tuple

$(H_0, H_1, ..., H_{n\text{-}1})$ where $H_k = \sum_{j=0}^{N-1} h_j e^{\frac{-i2\pi j}{N}}$ for $k = 0,1,...N\text{-}1$.

Parseval's Identity for DFT:

$$\underbrace{\sum_{\mathrm{j=0}}^{\mathrm{N\text{-}1}} |\mathrm{h_j}|^2}_{\|h\|^2} = \frac{1}{N} \sum_{k=0}^{N-1} |H_k|^2$$

Proof

$$\sum_{k=0}^{N-1} |H_k|^2 = \sum_{k=0}^{N-1} H_k \overline{H_k}$$

$$= \sum_{k=0}^{N-1} \left( \sum_{j=0}^{N-1} \sum_{l=0}^{N-1} h_j \overline{h_l} e^{\frac{-i2\pi jk}{N}} e^{\frac{i2\pi lk}{N}} \right)$$

Note(1) $\quad H_k = h_0 + h_1 e^{\frac{-i2\pi k}{N}} + h_2 e^{\frac{-i4\pi k}{N}} + ... + h_{N-1} e^{\frac{-i2\pi(N-1)k}{N}}$

Note(2) $\quad \overline{H_k} = \overline{h_0} + \overline{h_1} e^{\frac{i2\pi k}{N}} + \overline{h_2} e^{\frac{i4\pi k}{N}} + ... + \overline{h_{N-1}} e^{\frac{i2\pi(N-1)k}{N}}$

$$\sum_{k=0}^{N-1} |H_k|^2 = \sum_{j=0}^{N-1} \sum_{l=0}^{N-1} \sum_{k=0}^{N-1} h_j \overline{h_l} e^{\frac{-i2\pi jk}{N}} e^{\frac{i2\pi lk}{N}}$$

$$= \sum_{j=0}^{N-1} \sum_{l=0}^{N-1} h_j \overline{h_l} \sum_{k=0}^{N-1} e^{\frac{-i2\pi jk}{N}} e^{\frac{i2\pi lk}{N}} \quad \text{(changing the order of summation)}$$

$$= \sum_{j=0}^{N-1} \sum_{l=0}^{N-1} h_j \overline{h_l} \sum_{k=0}^{N-1} e^{\frac{i2\pi(l-j)k}{N}} \qquad *$$

Now consider the sum

$$\sum_{k=0}^{N-1} e^{\frac{i2\pi(l-j)k}{N}}.$$

If $l = j$, the sum is equal to $N$.

If $l \neq j$, let $w = e^{\frac{i2\pi(l-j)}{N}}$.

Note that $w \neq 1$ is an $N^{\text{th}}$ root of 1 ($w^N = 1$).

The sum is equal to $1 + w + w^2 + ... + w^{N-1}$.

$(1-w)(1+w+w^2+...+w^{N-1}) =$

$1+w+w^2+...+w^{N-1}$-$(w+w^2+...+w^{N-1}-w^N) = 1-w^N$.

Therefore $1+w+w^2+...+w^{N-1} = \dfrac{1-w^N}{1-w} = 0$, since $w^N = 1$.

$$\sum_{j=0}^{N-1} h_j \overline{h_j} N = N \sum_{j=0}^{N-1} \left|h_j\right|^2$$

It follows that $\sum_{j=0}^{N-1}\sum_{l=0}^{N-1} h_j \overline{h_l} \sum_{k=0}^{N-1} e^{\frac{i2\pi(l-j)k}{N}} = \sum_{j=0}^{N-1} h_j \overline{h_l} N = N \sum_{j=0}^{N-1} \left|h_j\right|^2$

and equation * implies that

$\sum_{k=0}^{N-1} \left|H_k\right|^2 = N \sum_{j=0}^{N-1} \left|h_j\right|^2$ q.e.d.

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