INTRODUCTION TO MATHCAD
Numerical Solution of Differential Equations

In this lesson you will earn how to solve both first and second order differential equations when working with mathematical models of mechanical systems. You will also learn how to solve two simultaneous nonlinear second order differential equations, as is required to analyze projectile motion when air resistance is included (the project).

Open a new worksheet in preparation for this lesson.

INTRODUCTION AND GENERAL CONCEPTS: The solution methods we investigate in this lesson are ***numerical***. That means that MathCAD *does not* solve the equations algebraically (exactly). Rather, it invokes the Runge-Kutta numerical method. The output from MathCAD is a matrix of numbers that represent the solution at a set of time steps or intervals. These points can be plotted and the dots can be joined to form a continuous line. **However, remember that the MathCAD solution is NOT the exact theoretical solution.** Usually, the difference is small and inconsequential, however sometimes the errors can be large. You should use good engineering judgment and solution verification to ensure the numerical answers are satisfactory.

The steps required to solve ordinary differential equations in MathCAD are:

1. Define the differential equation(s) in a form that MathCAD can 'understand.'

2. Define the initial conditions. These are required since the numerical solution has to know what values to start solving from.

3. Invoke the Runge-Kutta numerical procedure to solve the differential equation(s). MathCAD has several different implementations of the Runge-Kutta method. This lesson only considers one of them. You may find the "differential equations" topic in MathCAD's "Help" feature to be useful.

4. Graph the output and/or use the results.

FIRST ORDER DIFFERENTIAL EQUATION: As an example we will use the following simple equation for which the exact solution can easily be obtained by direct integration.

$$\frac{dy}{dt} = -5e^{-0.2t}$$

When subject to the initial condition, y(0) = 5, the exact solution is:

$$y(t) = 5 - 25\left(1 - e^{-0.2t}\right)$$

First step: Formatting the equation. For a single first-order differential equation, getting the equation into "MathCAD format" is relatively simple. Begin by selecting symbol for the independent and dependent variables. Here we use t and y respectively. We then define the differential equation using the symbol 'D(t,y)' to indicate the derivative. The first element of the function 'D' identifies the independent variable (t) and the second element identities the dependent variable (y). Enter the equation by creating the following definition:

$$D(t,y) := -5e^{-0.2t}$$

Note this is a simple statement of the differential equation with the first derivative expressed by the MathCAD function 'D(t,y)'.

Second step: Initial conditions. The initial condition we chose was y(0) = 5. Define the initial condition using the first element of a vector that is named for the dependent variable. Type

y[0:5

Note: It is best to leave the default setting of "ORIGIN~0" when solving differential equations.

Third step: Solve. The Runge-Kutta procedure is invoked using the MathCAD ***rkfixed*** function**.** Look it up in MathCAD's "Help". The function is called using the following syntax:

rkfixed(y, $t_{INITIAL}$, t_{FINAL}, n, D)

where

y = initial conditions for the dependent variable
$t_{INITIAL}$ = initial value of the independent variable
t_{FINAL} = final value of the independent variable
n = number of steps at which a solution is desired between $t_{INITIAL}$ and t_{FINAL}

Before we look at the answer, let's get the calculation done. For this exercise we will let the *rkfixed* function put the answer into a table we will call ***soln****.* Also, we will run the solution from time t = 0 up to t = 30. For now, we will have just 3 time steps in the solution. Type:

n:3
soln:rkfixed(y,0,30,n,D)
soln=

Fourth step: Output.

Note that ***soln*** is a matrix presented in tabular form. The first column (column zero) contains the values of the independent variable, t, and the second column contains the values of the solution for the dependent variable, y, at the respective values of t.

The following two methods will both extract the first column of ***soln*** into a new variable, **t2**, and the second column of ***soln*** into new variable **y2**.

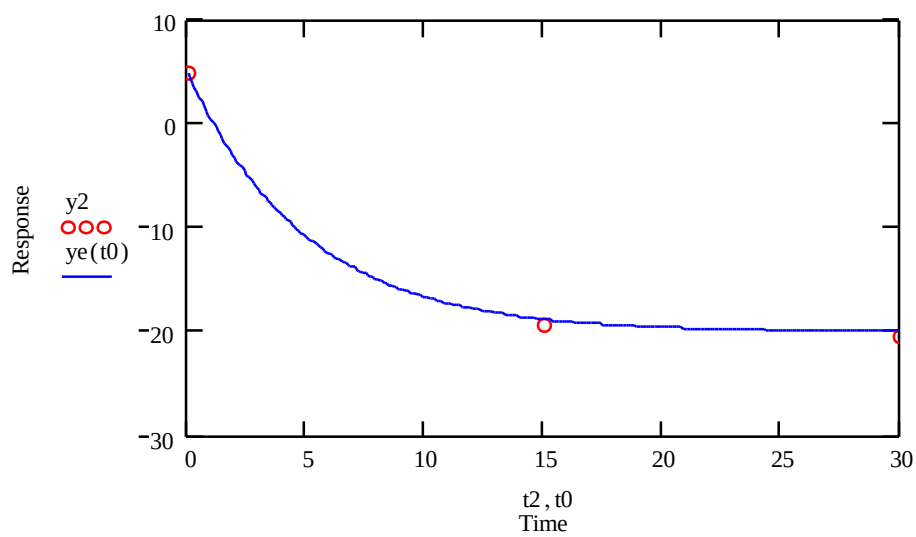
$$nr := \mathrm{rows}(soln)$$
$$t2 := \mathrm{submatrix}(soln, 0, nr - 1, 0, 0)$$
$$y2 := \mathrm{submatrix}(soln, 0, nr - 1, 1, 1)$$

$$i := 0..\mathrm{rows}(soln) - 1$$
$$t2_i := soln_{i,0}$$
$$y2_i := soln_{i,1}$$

Define variable function ***ye(t)*** to be the exact solution (given previously). Also define a range variable, t0, to have the values **0,0.1;30**

Plot both the exact and numerical results on a single graph. Add suitable x and y labels. Hide the legend, but not the arguments. Make the exact solution be a continuous line, and the approximate solution be just circles (no lines). Make your graph a suitable size (bigger!)

Your solution should look like this, with the exact algebraic (exact) solution showing as a blue line, and the numeric solution as red circles:



Note that using just 3 time steps was not sufficient. The red circles do not lie exactly on the blue line. Why?

Increase ‘n’ and see the difference in both accuracy and number of data points. Also notice that for large ‘n’, the ***soln*** table gets very big. If you **click on the table** you get a scroll bar so that you can move through the entire table.

Make your work presentable and, for n=50, save your work to file name `Mcad_DE_01`

<u>Another example.</u>

Obtain the numerical solution to the equation

$$\frac{dz}{dx} + 0.2z = 5$$

subject to the initial condition z(0) = 0. Calculate and plot your solution over the range $0 \le x \le 20$. Overlay the following exact solution and confirm both solutions are essentially the same.

$$z = 25\left(1 - e^{-x/5}\right)$$

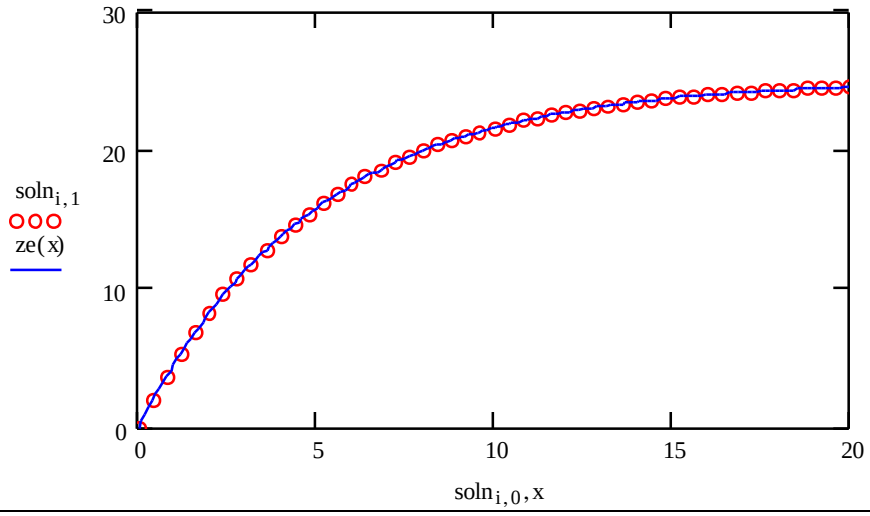
Note how the following MathCAD solution to this problem does not extract the required output from the ***soln*** table into different variables. Rather, it uses the table directly for plotting. To do this, first define ***i*** as an integer range variable that is used to ‘point’ to each row of the table. Then, the required column from the table is plotted using double-subscripts, e.g. $soln_{i,0}$ for the first (subscript-zero) column and $soln_{i,1}$ for the second (subscript-1) column.

$D(x,z) := 5 - 0.2 \cdot z \qquad z_0 := 0 \qquad n := 50 \qquad i := 0..n$

$soln := rkfixed(z, 0, 20, n, D)$

$x := 0, .1 .. 20$ x-axis for exact solution is a range variable

$ze(x) := 25 \cdot \left(1 - e^{\frac{-x}{5}}\right)$ exact solution



Make your work presentable and, for n=50, save your work to file name Mcad_DE_02

<u>SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS:</u> Now consider the second order differential equation:

$$a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = x(t)$$

Since the Runge-Kutta algorithm can only evaluate first order differential equations, MathCAD cannot directly solve a second order differential equation. The way we solve the equation is to recast the second-order equation into two first-order differential equations, and use MathCAD to solve these new equations simultaneously. Let's consider the previous example and define:

$$y1 = \frac{dy}{dt}$$

We can now use this equation to rearrange the second-order differential equation into two first-order equations

$$\frac{dy}{dt} = y1 \qquad \text{and} \qquad \frac{dy1}{dt} = -\frac{a_1}{a_2}y1 - \frac{a_0}{a_2}y + \frac{x(t)}{a_2}$$

These two equations are put into the "D" format by defining D(x,y) as a 2x1 column vector, with dy/dt as the first element, and d^2y/dt^2 as the second element.

We then need to set the initial conditions for both y and dy/dt. We also do this with a 2x1 column vector.

<u>Let's try an example.</u> Consider the equation:

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + 5y = 0$$

subject to the initial conditions y(0) = 4 and dy/dt(0) = 0. We will solve this equation over the range $0 \le t \le 10$.

<u>First step: Formatting the equation.</u>
Define y_1 = dy/dt and rewrite the equation as two first-order equations:

$$\frac{dy}{dt} = y_1$$

$$\frac{d^2y}{dt^2} = -\frac{dy}{dt} - 5y = -y_1 - 5y_0$$

Enter these equations in the "D" format for MathCAD in column matrix form as:

$$D(t, y) = \begin{bmatrix} y_1 \\ -y_1 - 5y_0 \end{bmatrix}$$

In this expression, the upper element is dy_0/dt and the lower element is dy_1/dt. Note the use of subscripts to identify the variables. Subscript-0 means the dependent variable (y), and subscript-1 means the first differential of the dependent variable (dy/dt).

<u>Second step: Initial Conditions.</u>
Set the initial conditions in MathCAD by defining them in a column vector. For this problem, the initial displacement, y, is 4, and the initial velocity, dy/dt, is zero**. Enter these conditions in a column matrix**. Note the use of another variable name, *yi*.

$$yi = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Third step: Solve.

The Runge-Kutta solver is invoked using 'rkfixed'. Let's put the solution into variable *'soln2'*. Recall, we wanted the range of times for the output to be $0 \le t \le 10$. Also, we will have 101 points (start point plus 100). **Enter the *'D'* and *'yi'* matrices**. Then type:

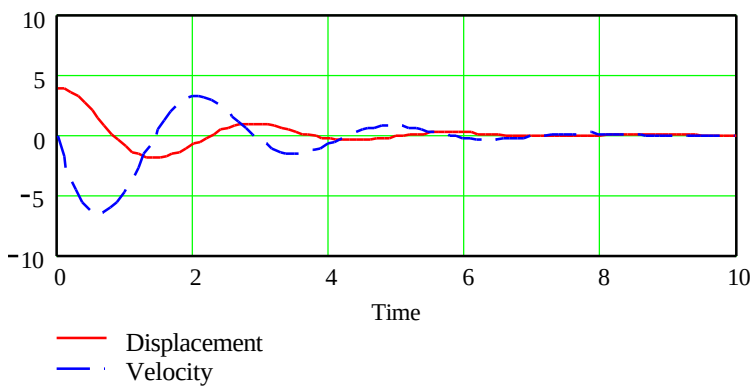
```
n:100
soln2:rkfixed(yi,0,10,n,D)
i:0;n
```

Fourth step: Output.

The output *'soln2'* is a three-column matrix. The first column contains the independent variable, t. The second column contains the solution for the dependent variable y, and the third column contains the solution for the dependent variable y_1 (i.e., dy/dt). Each successive row is a solution for that respective time step, ranging from t=0 to t=10 in 100 steps.

Overlay the plots of the solutions for y and dy/dt vs. time on the same graph. Make your output look like the following figure.

Hints: Auto Grid Lines for x-axis. Number of Grid Lines = 4 (not Auto Grid) for y-axis. Traces (Legend Labels, Line, Hide Arguments, don't Hide Legend), Labels (x-axis only).



Make your work presentable and formatted as a single-page enclosure.

For n=100, save your work to file name Mcad_DE_03

SIMULTANEOUS SECOND ORDER DIFFERENTIAL EQUATIONS:

Here comes the capstone in solving DE's! We will set up this problem using the example of a projectile including the effect of air resistance. This example is dealt with in detail during your project, so we only cover the highlights in this tutorial.

Note that we go right through the general solution for this problem first. Only near the end do we actually get on with an example. **So stop typing, and start reading!**

GOVERNING EQUATIONS OF MOTION:

The axis system we will use is x, y with x being the horizontal motion and y the vertical motion. For this exposition we will miss out drawing the FBDs (please don't tell you're your EM232 Dynamics instructor!) Applying Newton's Second Law we have two equations that govern the motion:

Horizontal motion: $$-F_d \cos(b) = ma_X = m\frac{d^2x}{dt^2}$$

Vertical motion: $$-F_d \sin(b) - mg = ma_Y = m\frac{d^2y}{dt^2}$$

In this formulation, b is the angle of the trajectory above the horizontal and F_d is the drag force cause by the air resistance. The drag force is calculated from a number of parameters:

C_D = coefficient of drag
r = air mass density
A = cross-sectional area of the projectile
v = speed of projectile $= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

and $$F_d = C_D \frac{1}{2} r v^2 A$$

For MathCAD we need to get the second-order equations as a set of first-order equations by finding:

$$\frac{d^2x}{dt^2} = fn\left(const, \frac{dx}{dt}, \frac{dy}{dt}\right)$$
$$\frac{d^2y}{dt^2} = fn\left(const, \frac{dx}{dt}, \frac{dy}{dt}\right)$$

Since the components of F_d depend on the speed of the projectile as well as the angle, we determine the following:

$$\cos(b) = \frac{v_X}{v} = \frac{\left(\frac{dx}{dt}\right)}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}$$

$$\sin(b) = \frac{v_Y}{v} = \frac{\left(\frac{dy}{dt}\right)}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}$$

Substitution and rearrangement of the above equations finally yields the second-order differential equations in the form we want:

$$\frac{d^2x}{dt^2} = \frac{-C_D\, r\, A}{2m}\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}^{1/2}\frac{dx}{dt}$$

$$\frac{d^2y}{dt^2} = -g - \frac{C_D\, r\, A}{2m}\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}^{1/2}\frac{dy}{dt}$$

<u>First step: Formatting.</u> Now it's time to get the equations ready for MathCAD. Let's define the following (note the use of subscripts for variable *p*):

$$C_O = \frac{C_d\, r\, A}{2m}$$

$$p_1 = \frac{dx}{dt} \qquad p_2 = \frac{d^2x}{dt^2}$$

$$p_3 = \frac{dy}{dt} \qquad p_4 = \frac{d^2y}{dt^2}$$

Now, MathCAD needs the 'D' format to be as follows:

$$D(t,p) = \begin{bmatrix} dx/dt \\ d^2x/dt^2 \\ dy/dt \\ d^2y/dt^2 \end{bmatrix}$$

Or using the variable defined above:

$$D(t,p)=\begin{bmatrix} p_1 \\ -C_O\left(\sqrt{p_1^2+p_3^2}\right)p_1 \\ p_3 \\ -g-C_O\left(\sqrt{p_1^2+p_3^2}\right)p_3 \end{bmatrix}$$

<u>Second step: Initial conditions.</u> As for the other numerical solutions, we put the initial conditions in a column vector. For the two second-order differential equations, we need 4 initial conditions. For this projectile problem, the initial conditions are:

Initial x-displacement; initial x-speed; initial y-displacement; initial y-speed

We will use *Vinit* as the initial launch speed, and *binit* as the initial launch angle. The initial conditions vector, *IC*, becomes:

$$IC=\begin{bmatrix} x_i \\ dx_i/dt \\ y_i \\ dy_i/dt \end{bmatrix} \quad \text{or} \quad IC=\begin{bmatrix} 0 \\ Vinit.\cos(b\,init) \\ 0 \\ Vinit.\sin(b\,init) \end{bmatrix}$$

<u>Third step: Solve.</u>
At last! Any easy step! We'll put the rkfixed output into variable *soln3*.

```
n:100
soln3:rkfixed(IC,0,10,n,D)
i:0;n
```

<u>Fourth step: Output.</u>
The output that appears in variable soln3 is in tabular (matrix) form. The columns of the output table are:

0	1	2	3	4
t	x	dx/dt	y	dy/dt

<u>Example.</u>

NOW it's time to start typing (but don't stop reading!) Open a new worksheet

For this example, we will look at scoring a home run in baseball. The ball has a diameter of 3 inches, and weighs 9 ozs. Let's assume the ball leaves the bat at 110 m.p.h. (or 161 ft/s) and the initial launch angle is 30° above the horizontal. Will the batter score a home run?

Rather than giving key-stroke instructions, look at the following MathCAD worksheet and see how the different input is entered. Also look at the final result.

First, set up the variables, etc in preparation for solving the equations.

PROJECTILE MOTION WITH AIR RESISTANCE

Equations are:

m d^2x/dt^2 = -Fd cos(b)

m d^2y/dt^2 = -mg - Fd sin(b)

Fd = Cd Af rho V^2/2

sin(b)=Vy/V

cos (b) = (Vx + Vw)/V

V = ((Vx+Vw)^2 +Vy^2)^0.5

Vx = dx/dt

Vy = dy/dt

tan (a) = Vy/Vx

$m := \frac{9}{16 \cdot 32.2}$ mass of ball (slugs)

$dia := \frac{3}{12}$ diameter (ft)

$Vinit := 110 \cdot \frac{5280}{3600}$ Launch speed (ft/s) $Vinit = 161.333$

$\beta init := 30 \cdot deg$ Launch angle - note conversion of degrees to radians

$\beta init = 0.524$

$A := \pi \cdot \frac{dia^2}{4}$ cross-sectional (frontal) area of ball (ft^2)

$\rho := 0.00238$ Mass density of air (slug/ft^3)

$Cd := 0.5$ Coefficient of Drag

$g := 32.2$ ft/s^2

$Co := Cd \cdot A \cdot \frac{\rho}{2 \cdot m}$ $Co = 1.672 \times 10^{-3}$

$$IC := \begin{pmatrix} 0 \\ Vinit \cdot \cos(\beta init) \\ 0 \\ Vinit \cdot \sin(\beta init) \end{pmatrix} \qquad IC = \begin{pmatrix} 0 \\ 139.719 \\ 0 \\ 80.667 \end{pmatrix}$$

Initial Conditions

$$D(t, p) := \begin{bmatrix} p_1 \\ -Co \cdot p_1 \cdot \sqrt{(p_1)^2 + (p_3)^2} \\ p_3 \\ -g - Co \cdot p_3 \cdot \sqrt{(p_1)^2 + (p_3)^2} \end{bmatrix}$$

Set up the "D" format

$n := 100$ $i := 0 .. n$ Solve for 100 time steps

Now solve the equations with air resistance. Then remove the air resistance and solve again. Finally, plot the two resulting projectile paths (one with, and one without air resistance).

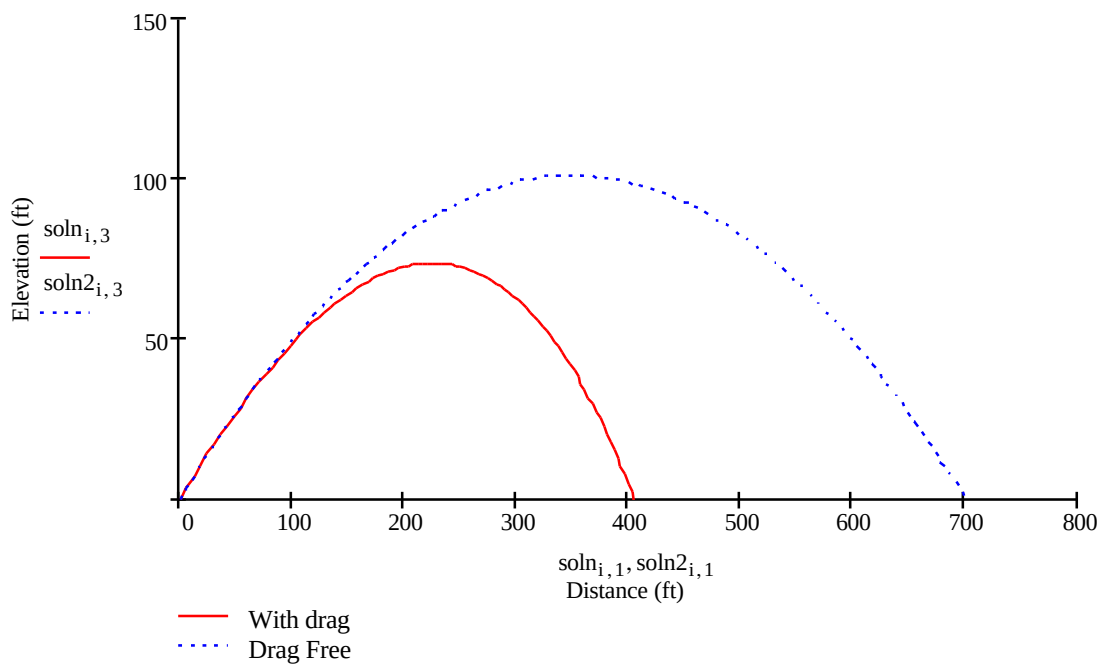
$soln := rkfixed(IC, 0, 4.54, n, D)$ Solve with air resistance

$Co := 0$ remove air resistance and resolve

$$D(t,p) := \begin{bmatrix} p_1 \\ -Co \cdot p_1 \cdot \sqrt{(p_1)^2 + (p_3)^2} \\ p_3 \\ -g - Co \cdot p_3 \cdot \sqrt{(p_1)^2 + (p_3)^2} \end{bmatrix}$$

Just copy&Paste from above - don't retype

$soln2 := rkfixed(IC, 0, 5.47, n, D)$



So, did the batter score a home run? His ball went just over 400 feet. Without air resistance his ball would have gone about 700 feet. So if air resistance makes such a big difference, why don't they play baseball on the moon?

Answer: Nobody would go because there's no atmosphere!

Save your work to file Mcad_DE_04. Then close the worksheet.

SUBMIT to your instructor:
Submit a printed copy of the worksheet you saved as file Mcad_DE_03.

Close MathCAD.
This introduction to solving differential equations is finished!