POISSON DISTRIBUTION

The Poisson distribution is often used in queuing theory. The Poisson distribution can be used to estimate the number of random events expected in a specific interval of time or space. Examples include: The number of phone calls arriving at an exchange during a one-hour window; the number of customers arriving at a checkout; the number of potholes in a one-mile long section of Interstate.

The following are the rules (properties) of the Poisson distribution:

> The probability of a single event happening does not change with time (or space)
>
> The probability of one event happening is independent of any other event happening.

Examples of when the properties are *not* followed, in which case the straightforward theory presented in this handout is not applicable, include:

> If a store has a single checkout (or the number of customers in all checkouts is combined), that's OK. But if there is more than one checkout, customers will naturally gravitate to the shorter lines. Thus, the independence of one person *arriving at the checkout* is not independent of previous events.
>
> The number of midshipmen using the Midstore increases during certain times of day, so the probability of someone going to the checkout is not constant throughout the day.

The probability of x events happening during the interval (of time or space) is given by:

$$P(x) = \frac{e^{-l}\, l^{x}}{x!}$$

where l is the expected or mean number of occurrences for that interval.

The variance of the Poisson distribution is

$$s^{2} = l$$

Just as with the Binomial distribution, if we wish to calculate the probability of *k or less* events, we have to calculate the summation:

$$\begin{aligned} P(x \le k) &= P(0) + P(1) + \ldots\ldots + P(k) \\ &= \sum_{i=0}^{k} \frac{e^{-l}\, l^{i}}{i!} \end{aligned}$$

EXAMPLE: Last year, 14,235 high school students applied to the Naval Academy. Assuming the probability of an individual applicant sending in their paperwork was constant throughout the 365-day year, calculate:

a) The expected number of applicants per day

b) The probability that there will be exactly the expected number of applicants on any one day

c) The probability that there will be less than the expected number of applicants on a randomly selected day.

d) The probability that there will be more than the expected number of applicants on a randomly selected day.

SOLUTION

a) $l = \frac{14235}{365} = 39$ applicants per day

b) $P(39) = \frac{e^{-l} l^{39}}{39!} = \frac{e^{-39} 39^{39}}{39!} = 0.063746 = 6.3746\%$

c)

$$P(x < 39) = P(0) + P(1) + ... + P(38)$$
$$= \sum_{i=0}^{38} \frac{e^{-l} l^{i}}{i!}$$
$$= \sum_{i=0}^{38} \frac{e^{-39} 39^{i}}{i!} = 0.478703 = 47.8703\%$$

(Using MathCAD to do the summation, of course)

d) We can either calculate $P(x > 39) = \sum_{i=40}^{\infty} \frac{e^{-l} l^{i}}{i!}$,

or the alternate approach:

$$P(x > 39) = 1 - P(x < 39) - P(x = 39)$$
$$= 1 - 0.478703 - 0.063746$$
$$= 0.457551 = 45.76\%$$

EXAMPLE: Testing of the cable used for a towed array shows that on average there are 2 defects per 300 feet of cable. The control and data acquisition system can tolerate 5 faults in a 500-foot long array.

What is the probability that a 500-foot tow manufactured from randomly selected cable will be unusable?

SOLUTION:

For a 500-foot length the expected number of faults is $l = \frac{2}{300} \times 500 = 3.3333$

$$
\begin{aligned}
P(5 \text{ or less}) &= \sum_{i=0}^{5} \frac{e^{-l} l^{i}}{i!} \\
&= \frac{e^{-l} l^{0}}{0!} + \frac{e^{-l} l^{1}}{1!} + \frac{e^{-l} l^{2}}{2!} + \frac{e^{-l} l^{3}}{3!} + \frac{e^{-l} l^{4}}{4!} + \frac{e^{-l} l^{5}}{5!} \\
&= 0.878833 = 87.88\%
\end{aligned}
$$

This calculation gives us the probability that a tow will be acceptable (5 or less faults). Therefore, we now calculate the probability of 6 or more faults as:

$$
\begin{aligned}
P(6 \text{ or more}) &= 1 - P(5 \text{ or less}) \\
&= 1 - 0.878833 = 0.121167 = 12.12\%
\end{aligned}
$$

PROBLEMS

Pois-1:

During an 8-hour day the main office in Bancroft receives 90 phone calls.

a) What is the expected rate of calls per hour?

b) What is the probability that there will be 5 or less calls in any period of one hour?

c) What is the expected rate of calls per minute?

d) You come on watch at 0800. What is the probability that in the first minute of your watch you will have to answer at least one call?

e) (advanced question) After how long can you reasonably expect to wait before you have to answer your third call?