MECHANICAL ENGINEERING DEPARTMENT
UNITED STATES NAVAL ACADEMY

EM423 - INTRODUCTION TO MECHANICAL VIBRATIONS

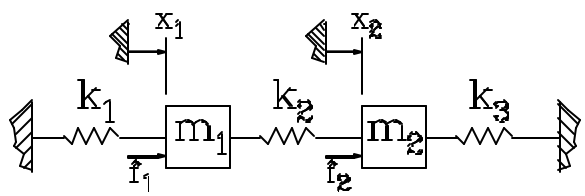
MANY DEGREES OF FREEDOM SYSTEMS

INTRODUCTION

This handout considers an undamped, lumped mass-spring system. The concepts are applicable to other Multi-Degree-of-Freedom (MDOF) systems, and hysteretic damping can be included for harmonic motion by letting the stiffness terms be complex. A previous handout determined the behavior of a two-degrees-of-freedom system as two simultaneous equations. This handout looks at the matrix solution of the two-DOF system, and extends it to general systems with many degrees of freedom. Experimental modal analysis, a method of determining the dynamic mathematical model of an existing structure, is discussed.

TWO DEGREES OF FREEDOM (2DOF)

Recall from the 2DOF handout that for the most general 2DOF system possible, the equations of motion can be written:



$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = f_1$$
$$m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = f_2$$

The matrix formulation of these equations is:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & (k_2 + k_3) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

or

$$[M][\ddot{x}] + [K][x] = [f]$$

We find the natural frequencies of this system by setting the excitation to zero. For unforced, harmonic motion at circular frequency w_r

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} e^{iwt}$$

hence

$$\begin{bmatrix} \left(k_1 + k_2 - m_1 w_r^2\right) & -k_2 \\ -k_2 & \left(k_2 + k_3 - m_2 w_r^2\right) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$$

The two circular natural frequencies are therefore the values of w_r which give the matrix a zero determinant:

$$\left| K - w_r^2 M \right| = 0$$

with $\quad [K] = \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & (k_2 + k_3) \end{bmatrix} \quad$ and $\quad [M] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$

or

$$\left(k_1 + k_2 - m_1 w_r^2\right)\left(k_2 + k_3 - m_2 w_r^2\right) - k_2^2 = 0$$

which is the same result determined using simultaneous equations in the 2DOF handout.

MANY DEGREES OF FREEDOM - UNDAMPED

For a system with many degrees of freedom, the equation of motion can also be shown in the matrix form:

$$[M][\ddot{x}] + [K][x] = [f(t)]$$

This course does not have time to fully investigate the matrices. The following are cursory discussions of the most important aspects of the matrices.

MASS MATRIX, *[M]*

For many engineering systems, the mass matrix can be formed as a diagonal matrix of the inertia elements. There are, however, some systems where the mass matrix is not diagonal. This course only considers diagonal mass matrix problems.

STIFFNESS MATRIX, *[K]*

The stiffness matrix has element k_{ij} in the ith row and jth column. It represents the force required at coordinate *i* while coordinate *j* has a unit displacement imposed *with all other displacements held at zero.* It is thus the *holding action* required. For linear elastic systems with small displacements reciprocity shows that $k_{ij} = k_{ji}$, and therefore the stiffness matrix is symmetric.

FLEXIBILITY MATRIX, *[H]*

The flexibility matrix has element h_{ij} in the ith row and jth column. It represents the displacement at coordinate *i* due to unit force applied *only* to coordinate *j* with all other coordinates allowed to move freely. For linear elastic systems with small displacements

reciprocity shows that $h_{ij} = h_{ji}$, and therefore the flexibility matrix is symmetric. For the engineering systems we consider here, the following relationship exists:

$$[H] = [K]^{-1}$$

The experimental significance and difference of the stiffness and flexibility matrices is discussed in class.

NATURAL FREQUENCIES AND MODE SHAPES

The natural frequencies and mode shapes (displaced shapes at resonance) are found by solving the equation of motion with *f(t)* = zero and substituting $[x] = [X]\, e^{iwt}$.

$$[M][\ddot{x}] + [K][x] = [f(t)]$$

hence

$$\left[K - w_r^2 M\right][X] = 0$$

or

$$[M]^{-1}[K][X] = w_r^2 [X]$$

This is an eigenvalue problem! The eigenvalues represent the (circular natural frequencies)^2, and the eigenvectors represent the mode shapes. Combined, they are called the *eigenstructure.* How are you going to find the eigenstructure? How about using a commercial computer package, such as Mathcad or Matlab?

MODAL and FREQUENCY MATRICES

The eigenstructure can represent a lot of information, especially for a structure with many degrees of freedom. Instead of giving long lists of numbers, the eigenstructure is combined and normally presented in two matrices. These matrices are the FREQUENCY MATRIX *[W]*, which is a diagonal matrix of the eigenvalues (circular natural frequencies), and the MODAL MATRIX *[F]*, which is a full matrix whose **columns** are the eigenvectors (mode shapes).

Frequency matrix, *[W]*:

$$[\Omega]^2 = \begin{bmatrix} w_1^2 & & & & \\ & w_2^2 & & & \\ & & .. & & \\ & & & .. & \\ & & & & w_N^2 \end{bmatrix}$$

Modal matrix, *[F]*:

$$[\Phi] = \begin{bmatrix} {}_1f_1 & {}_2f_1 & .. & .. & {}_Nf_1 \\ {}_1f_2 & {}_2f_2 & .. & .. & .. \\ {}_1f_3 & .. & .. & .. & .. \\ .. & .. & .. & .. & .. \\ {}_1f_N & .. & .. & .. & {}_Nf_N \end{bmatrix}$$

ORTHOGONALITY & NORMALIZATION

It can be shown that the eigenvectors (mode shapes) are orthogonal with respect to both the mass and stiffness matrices. This means:

$$[\Phi]^T [M][\Phi] = m_P [I]$$
$$[\Phi]^T [K][\Phi] = m_P [\Omega]^2$$

where m_P is the *modal mass*. The modal mass is an arbitrary number and is used to scale, or normalize, the eigenvectors.

DECOUPLING THE EQUATIONS

The ($N \times N$) matrix equation of motion represents N separate equations, and each equation includes several different spatial coordinates. This means the equations are **coupled**, and mathematically are complex to solve. Decoupling means applying a coordinate transformation such that each equation only has one independent coordinate. The new coordinates are called the **modal (or principal) coordinates**.

The modal coordinates relate directly to the natural frequencies of a structure, and there is one coordinate for each resonance. If we consider the case for unit modal mass ($m_P = 1$), the equations of motion can be decoupled as follows:

$$[M][\ddot{x}] + [K][x] = [f(t)]$$

let

$$[x] = [\Phi][a]$$

substituting give

$$[M][\Phi][\ddot{a}] + [K][\Phi][a] = [f(t)]$$

Pre-multiplying by the transpose of the modal matrix gives:

$$[\Phi]^T [M][\Phi][\ddot{a}] + [\Phi]^T [K][\Phi][a] = [\Phi]^T [f(t)]$$

Now we apply the restrictions of orthogonality and simple harmonic motion to get:

$$-w^2 [a] + [\Omega]^2 [a] = [\Phi]^T [f(t)] = [f_G(t)]$$

or, rearranging:

$$\left[\Omega^2 - w^2 I\right][a] = \left[f_G(t)\right]$$

where $f_G(t)$ is called the *Generalized Force Vector.*

Since all the quantities in the equation are either diagonal matrices or column vectors, the matrix result represents *N* independent equations:

$$\begin{aligned}
\left(w_1^2 - w^2\right) a_1 &= f_{G_1}(t) \\
\left(w_2^2 - w^2\right) a_2 &= f_{G_2}(t) \\
&\cdots\cdots\cdots\cdots\cdots \\
\left(w_N^2 - w^2\right) a_N &= f_{G_N}(t)
\end{aligned}$$

Each of these equations can be solved as if it where a single degree of freedom system. The final result for the displacement of the structure is determined from the previous substitution:

$$[x] = [\Phi][a]$$

SINGLE POINT EXCITATION AND RESPONSE

The previous result can be applied to linear MDOF systems with harmonic excitation at any or all coordinates. However, when conducting a vibration test, one common method is to excite the structure at only one point. For single point excitation at coordinate # j, the spatial force vector is all zeroes except for the entry that represents the excitation coordinate:

$$\left[f(t)\right] = \begin{bmatrix} 0 \\ .. \\ .. \\ f_j \\ .. \\ 0 \end{bmatrix}$$

If we now further restrict our interest only to the response at a single coordinate #i, it can be shown that for viscous damping:

$$\frac{X_i}{F_j} = \sum_{r=1}^{N} \frac{{}_r f_i \cdot {}_r f_j}{m_r \left\{ w_r^2 - w^2 + 2iz_r w_r w \right\}}$$

This equation is the summation of *N* single degree of freedom systems, and shows that the MDOF behavior of a structure can be modeled by summing (superimposing) the response of many SDOF oscillators.

For a given structure, we need to be able to calculate the properties of these SDOF oscillators. Experimentally we do this using a technique called *MODAL ANALYSIS.*

EXPERIMENTAL MODAL ANALYSIS

Experimental modal analysis is a method of determining a mathematical model that has the same dynamic properties as an existing structure. It is a state-of-the-art method, in that it is only in relatively recent history that is has been accepted as a fully functional method. One of the first useable references was: *Use of Vectors in Vibration Measurement and Analysis*. Kennedy & Pancu. Journal of the Aeronautical Society, 1947

1960-70	Theoretical algorithms were limited by functionality of the hardware.
1970-1990	Hardware improvements (especially good quality digital systems) meant measured data gave a better understanding of the theory and required algorithms.
1990s	A fully functional, general purpose, experimental tool.
today	New applications . . . . . .? ? ?

WHAT DOES MODAL ANALYSIS GIVE?

Modal analysis essentially gives the properties of the many SDOF oscillators which, when combined, have the same dynamic behavior as the structure. This information is presented as the natural (resonant) frequencies, damping (energy loss or dissipation), and deflected shapes at resonance (mode shapes) of the structure. Since these are system properties, *the method used for the experiments does not usually change or restrict the use of the derived model.*

WHAT CAN WE USE THE MODAL MODEL FOR?

There are numerous applications. Some of the most common are:

- Finite Element model validation
- Transmissibility estimation
- Non Destructive Examination (NDE)
- Health Monitoring
- Quality Control
- Prediction of transient structural response
- Damage detection

COMPUTER DEMONSTRATION

This demonstration shows the results of a modal analysis. Use this page for your own notes.

<u>MODAL CONSTANT</u> (called *modal residue* in some commercial software)
The modal constant is a complex number that defines the relative motion (magnitude and phase) between two coordinates on a structure at resonance. There is a different modal constant for every combination of excitation and response coordinates, and for every natural frequency.

<u>Example.</u> For a structure with a mesh of 56 coordinates, with single point response and excitation at all 56 coordinates, an analysis of 11 natural frequencies will determine a maximum of $56^2 \times 11 = 34{,}496$ modal constants. If we assume reciprocity, we can reduce this to approximately 616 different constants.

From the series:

$$\frac{X_i}{F_j} = \sum_{r=1}^{N} \frac{{}_r f_i \cdot {}_r f_j}{m_r \left\{ w_r^2 - w^2 + 2iz_r w_r w \right\}}$$

the modal constant, ${}_r A_{ij}$, is defined as the terms

$${}_r A_{ij} = \frac{{}_r f_i \cdot {}_r f_j}{m_r}$$

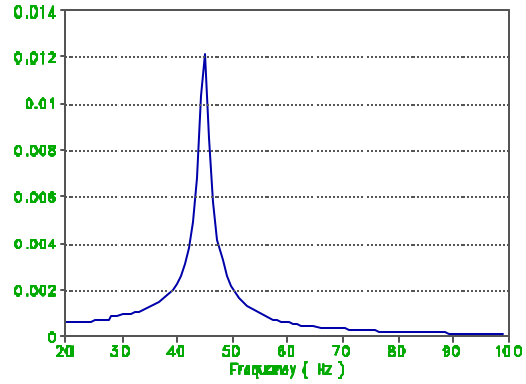
Which makes the series become:

$$\boxed{\frac{X_i}{F_j} = \sum_{r=1}^{N} \frac{{}_r A_{ij}}{\left\{ w_r^2 - w^2 + 2iz_r w_r w \right\}}}$$

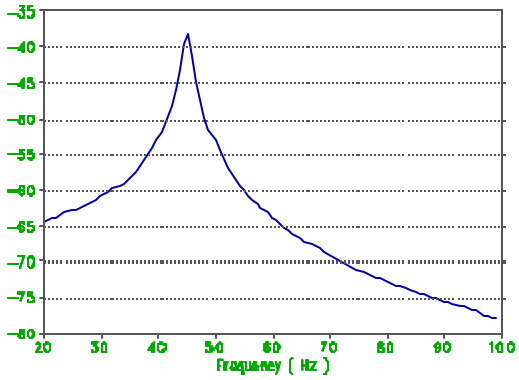
This is the equation to which we curve fit the experimental data. It can be shown from the normalization procedure that the modal constant is <u>independent</u> of the choice of modal mass.

DATA REQUIREMENTS FOR A MODAL ANALYSIS - FREQUENCY RESPONSE FUNCTIONS

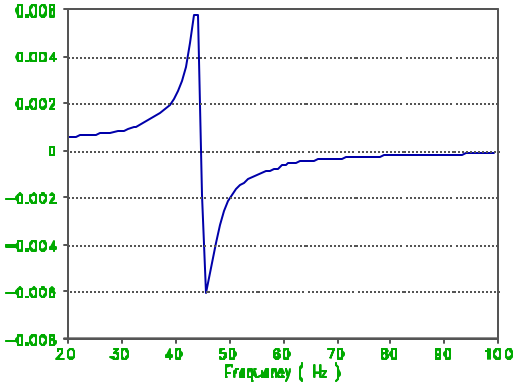
These graphs are for a single degree of freedom system.



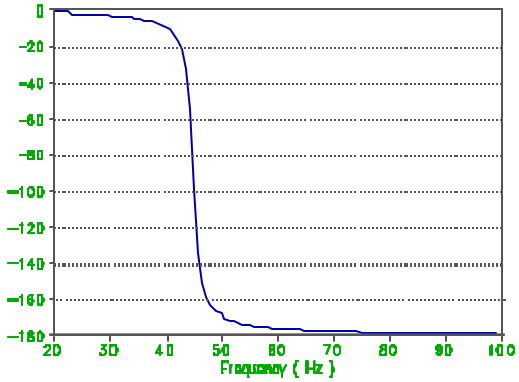
Linear magnitude vs. frequency



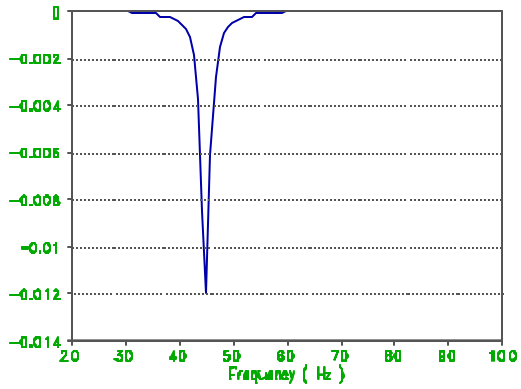
Log magnitude (dB) vs. frequency



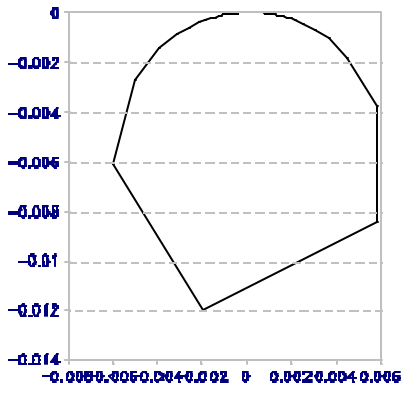
Real vs. frequency



Phase vs. frequency

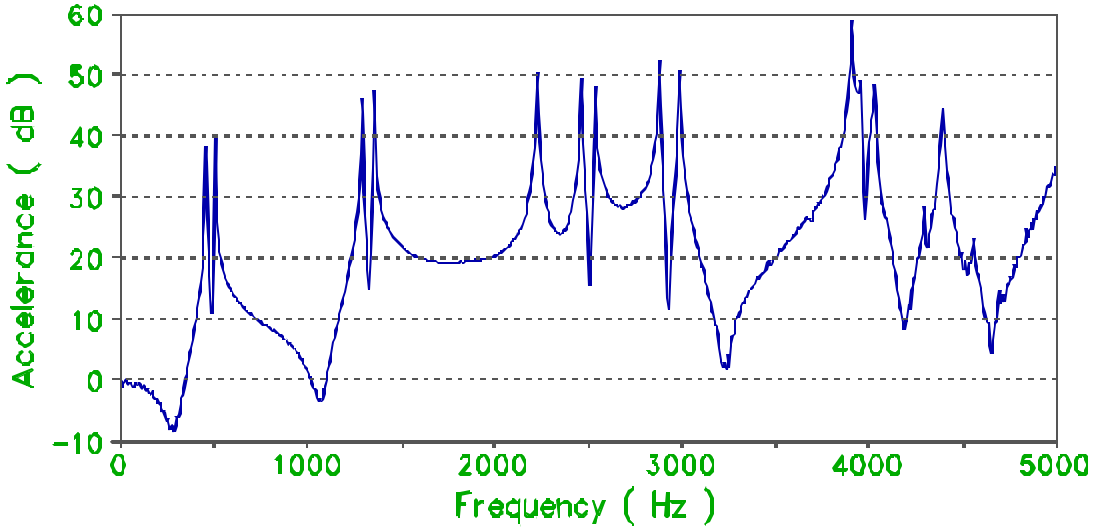


Imaginary vs. frequency



Nyquist (Real vs. Imaginary)

The following is an example of a measured FRF (dB vs. frequency). This FRF was measured from a GRP cylinder.



<u>Discussion points:</u>

Scaling	logarithmic v. linear (FRF and frequency)
Complex data	real, imaginary, modulus, phase
Coherence	
Point and transfer functions	

FOURIER TRANSFORMS

A Fourier Transform converts a **time signal** into a ***frequency* signal**. Both the signal and it's transform hold the **same information,** but present it in a different way. Fourier Analysis assumes **infinitely long, continuous** signals in both the time and frequency domains.

The transform of a harmonic time sine wave at frequency ω is an impulse in the frequency domain at frequency ω.

An impulse in the time domain has energy at all frequencies in the frequency domain.

In general, the “wider” the time pulse, the "narrower" the frequency "pulse". This means that if we wish to use impulsive excitation, and target low frequencies, we need a wide (i.e. long) time pulse. Conversely, if we wish to target higher frequencies, we need a much shorter time pulse.

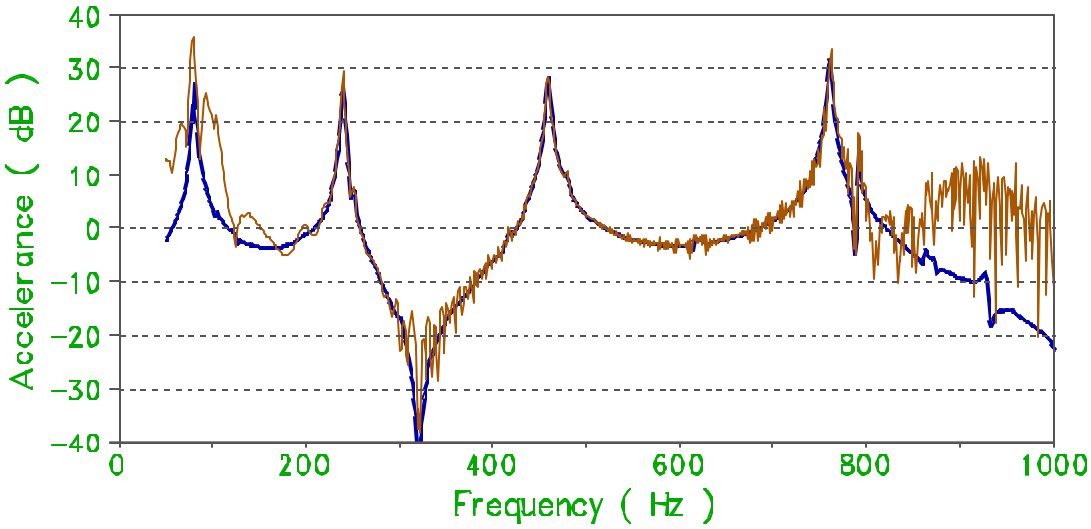
DISCRETE FOURIER TRANSFORM

Experimental data is **discrete and truncated.** Discrete time data transforms to periodic frequency data. Truncated (finite) time data transforms to discrete frequency data.

Discrete Fourier theory of truncated series assumes the time signal repeats periodically. Therefore the differences in the signal between the start and end of the time period can cause problems when transformed. These problems, if not taken into account, can severely degrade the data.

EXAMPLE OF TRUNCATION EFFECTS

There are many ways the edge or end effects can manifest themselves. The following is an example of degraded data. The “smooth” line is a FRF measured from a structure (the same GRP cylinder as used for the previous example). The “spiky” line is supposed to be the same FRF, but the truncation effects were allowed to be a problem.



WINDOWING

The way of reducing the truncation effects is to **weight the time data**. This usually ensures the time signals are zero, or very close to zero, at both the start and end of the time period, and reduces the problem. This process is called **WINDOWING**. The theory of windowing can give an indication of which type of window is best for which application. However, a good "rule of thumb" is to try different windows, and select the one that gives the best results for any one application. EXPERIMENT! In class, we will go through a better method that optimizes the window for your particular structure and experiment.

WINDOWING (CONTINUOUS, RANDOM SIGNALS)

When the excitation signal and resulting response are continuous random, both the beginning and end of the time signals have to be tapered. Examples of commonly used windows are:

HANNING (cosine^2)
HAMMING (Hanning on a rectangular pedestal)
GAUSSIAN

WINDOWING (TRANSIENT, IMPULSIVE SIGNALS)

The time signals are zero at the start, since the system is quiescent. The **response signal** is generally exponential. It needs "tapering off" toward the end. A good window for this type of data is **exponential**. Choose the time constant such that the signal is very close to zero at the end of the time period. If the time constant is too short, the real signal will be "lost" in the digitization noise. If the constant is too long, the window is ineffective, and end effects will be present. If an exponential window is not available, use a **rectangular** window (i.e. no window).

The **force signal** is impulsive, and zero everywhere except during the impact itself. The best window for this type of data is a square window around the impulse. If this is not available on the hardware, an exponential window (similar to the response window) may be adequate.

STEADY STATE EXCITATION

For this style of test, the structure is excited continuously at one frequency. After a short delay period, the response is steady state (i.e. not changing with time). The excitation and response can be measured very accurately, with digital filters at the exact excitation frequency. This gives a very high signal to noise ratio, and is particularly suitable where signals are small and/or the environment is noisy (acoustics or background vibration). The excitation frequency is incremented, and the process repeated. In this way, the transfer function is measured over the frequency range of interest.

No time window is necessary, because the data are not being Fourier Transformed.

ACCURACY EXPECTATIONS OF MODAL ANALYSIS RESULTS

The accuracy of modal data is highly dependent on the vibration characteristics of the structure, the accuracy of the data, and signal noise during measurement. The following aspects are discussed in class.

Natural Frequencies

Frequency Resolution

Damping

Too much, or too little?

Most appropriate mathematical model?

Modal Constants

Modal Assurance Criterion table (MAC) (based on orthogonality)

Consistency of animated shapes

MODAL TEST PROCEDURE

Choose an appropriate test method (transient, random, steady state)

Identify the spatial mesh and coordinates

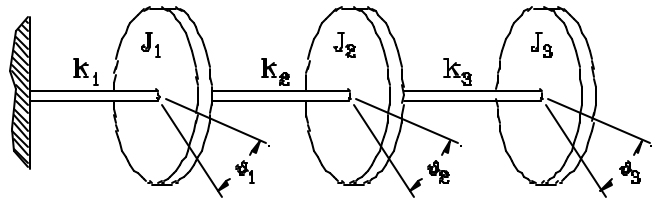
Measure transfer functions, referenced to either fixed excitation or fixed response.

Curve fit the data

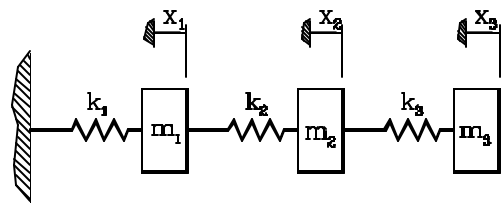
Verify the accuracy and reliability of the results.

ASSIGNMENTS

1. Determine the stiffness and flexibility matrices for the given system. Verify that the flexibility matrix is the inverse of the stiffness matrix.



2. Determine the stiffness and flexibility matrices for the given system, and write the equations of motion in the following matrix form. You may reference your solution to the previous question.



$$[M][\ddot{x}]+[K][x]=[f]$$

3. (extra credit) The normal modes of a 3-DOF system with $m_1 = m_2 = m_3 = m$ are given as:

$$[f_1]=\begin{bmatrix}0.737\\0.591\\0.328\end{bmatrix};\quad [f_2]=\begin{bmatrix}-0.591\\0.328\\0.737\end{bmatrix};\quad [f_3]=\begin{bmatrix}0.328\\-0.737\\0.591\end{bmatrix}$$

Verify the orthogonality of these mode shapes with respect to the mass matrix.

SOLUTIONS

1. Determine the stiffness and flexibility matrices for the given system. Verify that the flexibility matrix is the inverse of the stiffness matrix.

$$[M]=\begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix}; \quad [K]=\begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_2 & (k_2+k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$

$$[H]=\begin{bmatrix} \dfrac{(k_1k_3+k_3^2)}{\Delta} & \dfrac{k_2k_3}{\Delta} & \dfrac{k_2k_3}{\Delta} \\ \dfrac{k_2k_3}{\Delta} & \dfrac{(k_1k_3+k_2k_3)}{\Delta} & \dfrac{(k_1k_3+k_2k_3)}{\Delta} \\ \dfrac{k_2k_3}{\Delta} & \dfrac{(k_1k_3+k_2k_3)}{\Delta} & \dfrac{(k_1k_3+k_1k_3+k_2k_3)}{\Delta} \end{bmatrix}; \quad \text{with } \Delta=k_1k_2k_3$$

$$\text{verify: } [H][K]=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Determine the stiffness and flexibility matrices for the given system, and write the equations of motion in the following matrix form. You may reference your solution to the previous question.

Solution note: Problems 1 and 2 are dynamically the same system. The only difference is that in problem #1, we have “torsional springs and inertias,” whereas in problem #2 we have “linear springs and masses.” Thus, the units for the *k*’s and *m*’s are different (N-m/rad and N/m; kg-m^2 and kg), but the form of the equations is the same.

$$[M][\ddot{x}]+[K][x]=[f]$$

$$[M]=\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}; \quad [K]=\begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_2 & (k_2+k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$

$$[H]=\begin{bmatrix} \dfrac{(k_1k_3+k_3^2)}{\Delta} & \dfrac{k_2k_3}{\Delta} & \dfrac{k_2k_3}{\Delta} \\ \dfrac{k_2k_3}{\Delta} & \dfrac{(k_1k_3+k_2k_3)}{\Delta} & \dfrac{(k_1k_3+k_2k_3)}{\Delta} \\ \dfrac{k_2k_3}{\Delta} & \dfrac{(k_1k_3+k_2k_3)}{\Delta} & \dfrac{(k_1k_3+k_1k_3+k_2k_3)}{\Delta} \end{bmatrix}; \quad \text{with } \Delta = k_1k_2k_3$$

$$\text{verify: } [H][K]=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now put the equations in the required form:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix}+\begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_2 & (k_2+k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

3. (extra credit) The normal modes of a 3-DOF system with $m_1 = m_2 = m_3 = m$ are given as:

$$[f_1]=\begin{bmatrix} 0.737 \\ 0.591 \\ 0.328 \end{bmatrix}; \quad [f_2]=\begin{bmatrix} -0.591 \\ 0.328 \\ 0.737 \end{bmatrix}; \quad [f_3]=\begin{bmatrix} 0.328 \\ -0.737 \\ 0.591 \end{bmatrix}$$

Verify the orthogonality of these mode shapes with respect to the mass matrix.

The solution to extra credit questions is available from your instructor.