MECHANICAL ENGINEERING DEPARTMENT
UNITED STATES NAVAL ACADEMY

EM423 - INTRODUCTION TO MECHANICAL VIBRATIONS

TWO DEGREES OF FREEDOM SYSTEMS

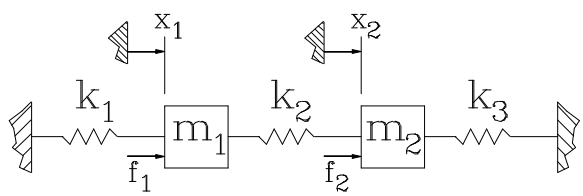
INTRODUCTION

This handout considers an undamped, lumped mass-spring system. The concepts are applicable to other two-degrees-of-freedom (2DOF) systems, and hysteretic damping can be included for harmonic motion by letting the stiffness terms be complex in the final equations. The handout determines the behavior of a 2DOF as two simultaneous equations. The special case of a tuned vibration absorber is discussed.

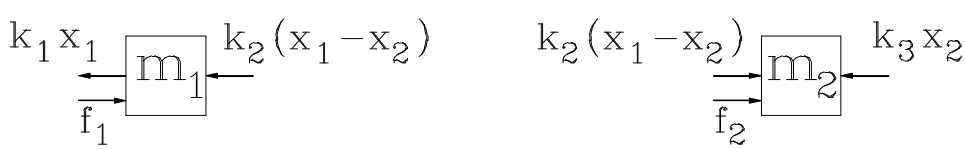
Extending this work to include systems with many degrees of freedom is left to a later handout.

TWO-DEGREES-OF-FREEDOM-SYSTEM (2DOF)

Consider the 2DOF shown. This is the most general discrete mass/spring system possible, since it includes connectivity (springs) between all the coordinates, and forcing at both coordinates.



Free Body Force Diagrams (FBDs) for the masses are:



Applying Newton's Second Law:

$$\text{for mass } m_1: \quad -k_1x_1 - k_2\left(x_1 - x_2\right) + f_1 = m_1\ddot{x}_1$$
$$\text{for mass } m_2: \quad k_2\left(x_1 - x_2\right) - k_3x_2 + f_2 = m_2\ddot{x}_2$$

from which the two equations of motion can be determined:

$$\text{for mass } m_1: \quad m_1\ddot{x}_1 + \left(k_1 + k_2\right)x_1 - k_2x_2 = f_1$$
$$\text{for mass } m_2: \quad m_2\ddot{x}_2 - k_2x_1 + \left(k_2 + k_3\right)x_2 = f_2$$

First we consider the unforced case ($f_1 = f_2 = 0$), with harmonic response:

$$x_1 = X_1 e^{iw_n t} \quad \text{and} \quad x_2 = X_2 e^{iw_n t}$$

Substituting these solutions into the two equations of motion yields two simultaneous equations of motion:

$$\left(k_1 + k_2 - w_n^2 m_1\right) X_1 - k_2 X_2 = 0$$
$$\left(k_2 + k_3 - w_n^2 m_2\right) X_2 - k_2 X_1 = 0$$

These equations have three unknowns: w_n, X_1 and X_2. Although we cannot solve for all three unknowns, we can solve the equations for w_n, and the ratio (X_2 / X_1).

From the first equation:

$$\left(\frac{X_2}{X_1}\right) = \frac{\left(k_1 + k_2 - w_n^2 m_1\right)}{k_2}$$

From the second equation:

$$\left(\frac{X_2}{X_1}\right) = \frac{k_2}{\left(k_2 + k_3 - w_n^2 m_2\right)}$$

Combining these equations to eliminate (X_2 / X_1) yields:

$$\left(k_1 + k_2 - w_n^2 m_1\right)\left(k_2 + k_3 - w_n^2 m_2\right) - k_2^2 = 0$$

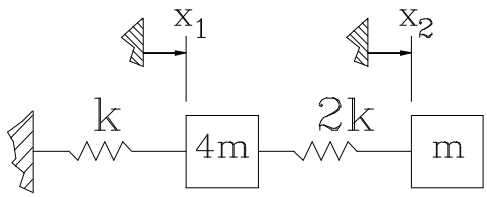
This equation is called the *frequency equation,* since it can be used to calculate the TWO natural frequencies of the system. The solutions are usually given the individual symbols w_1 and w_2, and collectively w_r (r = 1, 2).

For each w_r solution there is a different ratio of (X_2 / X_1). These ratios are called the *mode shapes,* and give the relative motion of the masses at resonance. They are usually shown:

$$\left(\frac{X_2}{X_1}\right)_1 \quad \text{and} \quad \left(\frac{X_2}{X_1}\right)_2$$

The system (equations) can also be solved using matrix methods. This is discussed in the Multi Degrees Of Freedom (MDOF) handout.

<u>Example:</u> Find the natural frequencies and mode shapes for the following system:



The two simultaneous equations obtained from the FBDs are:

$$\left(3k - w_r^2 4m\right) X_1 - 2kX_2 = 0$$

$$\left(2k - w_r^2 m\right) X_2 - 2kX_1 = 0$$

from which

$$4m^2 w_r^4 - 11kmw_r^2 + 2k^2 = 0$$

$$w_{1,2}^2 = \frac{11k}{8m} \pm \frac{k}{8m}\sqrt{89}$$

hence

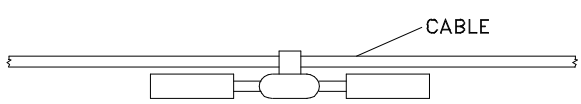
$$w_1^2 = 0.1957\frac{k}{m} \qquad w_2^2 = 2.554\frac{k}{m}$$

$$\left(\frac{X_2}{X_1}\right)_1 = 1.108 \qquad \left(\frac{X_2}{X_1}\right)_2 = -3.608$$

<u>THE VIBRATION ABSORBER</u>

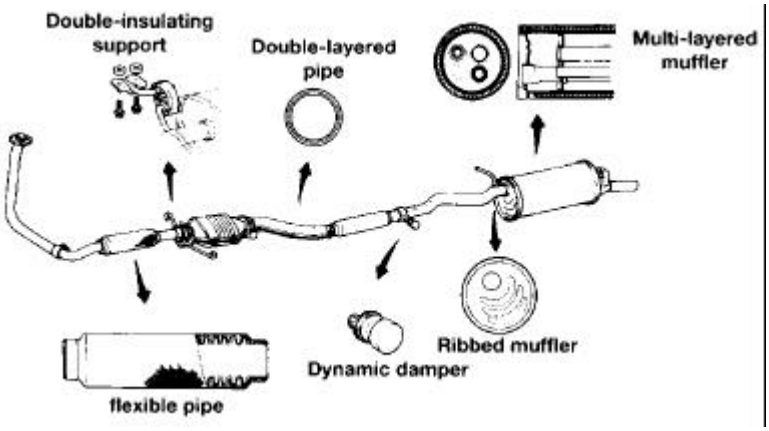
An approach to protecting a device from steady-state disturbances at constant frequency is to use a *vibration absorber.* The absorber consists of a *secondary mass-spring* combination added to the *primary* device to protect it from vibrating. Essentially this converts a single-degree-of-freedom system into a two-degrees-of-freedom system. Designing a system like this means we can have some control over the vibration of a component. Despite the name vibration absorber, we are NOT absorbing vibration. Rather, we are changing the impedance of the original system so that it no longer resonates at the excitation frequency.

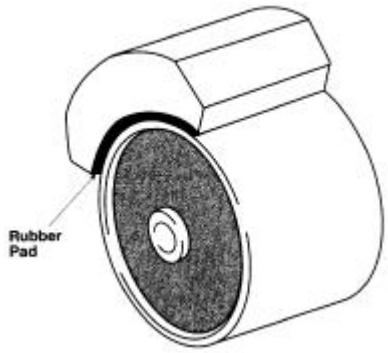
Devices that control vibration this way are found in a variety of applications. One common application is to reduce the vibration of high-tension power lines.



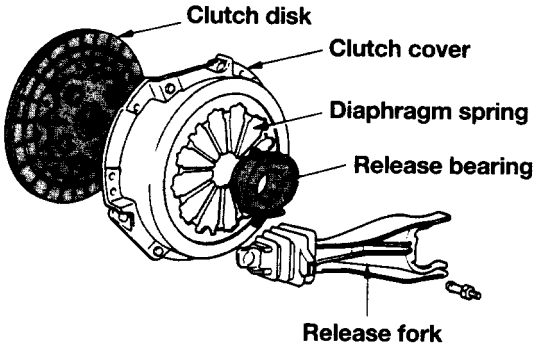
Stockbridge Damper

The automotive industry also uses vibration absorbers. For example[1], engine mounts and exhaust systems have secondary masses attached to act as vibration absorbers.





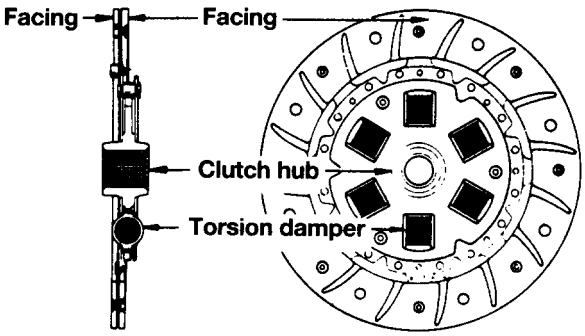
Other common applications include reciprocating power tools, electric razors, grinding wheels, musical instruments, and a variety of military applications. While this handout only considers a linear absorber (which is tuned to a single frequency), there are absorbers designed on similar principles to reduce the torsional vibration in internal combustion engines and the like. These can be designed to be self-tuning to the (variable) running speed or harmonics thereof.



Clutch Assembly

Fig. 3-20

The following exposition assumes damping is negligible. We can include hysteretic damping in the final equations by letting the stiffness term be complex. Generally, <u>increasing</u> damping <u>reduces</u> the effectiveness of the absorber.



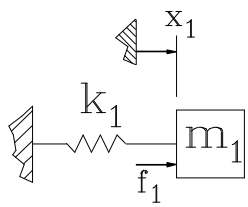
Clutch Disc

Fig. 3-21

[1]Automotive examples and figures are for Toyota vehicles

SYSTEM WITH ONE DEGREE OF FREEDOM (SDOF) - review

This is a brief review of the theory for a SDOF with harmonic excitation. See the previous handouts for full details.



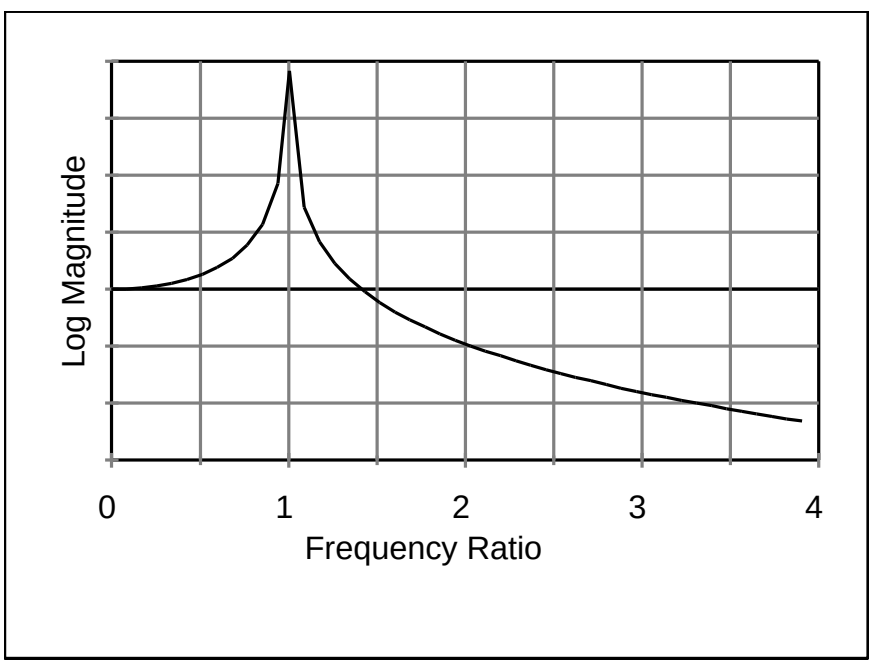
The system has a circular natural frequency given by $w_n = (k_1/m_1)$. The equation of motion with forcing is:

$$m_1\ddot{x}_1 + k_1 x_1 = f_1 = F_1 e^{iwt}$$

and the response is also harmonic, with $x_1 = X_1 e^{iwt}$ and:

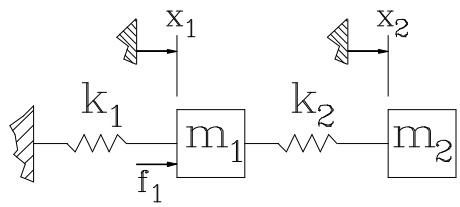
$$X_1 = \frac{F_1}{m\left(w_n^2 - w^2\right)}$$

If the excitation frequency, w, is close to the natural frequency, w_n the system will resonate - we get a very large response. The system will vibrate at ANY excitation frequency, but the amplitude of response is largest when the excitation frequency is close to the natural frequency.



MAKING THE VIBRATION ABSORBER

The concept of the vibration absorber is that we want to reduce the motion of the mass to zero. To do this, first let us modify the SDOF to make it a 2DOF, as shown. Note that there is no forcing on the added mass, m_2. The only forcing is the original forcing on mass m_1.



The equations of motion for the masses are:

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = f_1 = F_1e^{iwt}$$
$$m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = 0$$

Substituting $x_1 = X_1e^{iwt}$ and $x_2 = X_2e^{iwt}$ yields two simultaneous equations:

$$X_1 = \frac{(k_2 - m_2w^2)}{\Delta}F_1$$
$$X_2 = \frac{k_2}{\Delta}F_1$$
$$\text{with} \quad \Delta = (k_1 + k_2 - m_1w^2)(k_2 - m_2w^2) - k_2^2$$

These equations define the dynamics of our original SDOF after it has been modified by attaching the secondary mass/spring system. The extra mass and spring are the absorber. How does the absorber work? Ideally, we completely want to stop the vibration of our primary mass, m_1. We can do this by setting $X_1 = 0$ in the above equations. This yields:

$$w = \sqrt{\frac{k_2}{m_2}}$$

That is, if the natural frequency of the added mass-spring system BY ITSELF is the same as the EXCITATION FREQUENCY, the primary mass will stop moving. What this means is that we can *tune* an absorber to a single excitation frequency.

The X_2 equation can be rearranged to find out how much the secondary mass will vibrate:

$$X_2 = \frac{-F_1}{k_2}$$

This indicates that providing the original system is being driven by a ***force*** input, the motion of the added mass is bounded. It does not go infinite, even though it is being excited at a frequency that matches its original natural frequency.

If the undesirable vibration was at the natural frequency of the original SDOF, we can even target this frequency, by choosing:

$$\sqrt{\frac{k_2}{m_2}} = \sqrt{\frac{k_1}{m_1}}$$

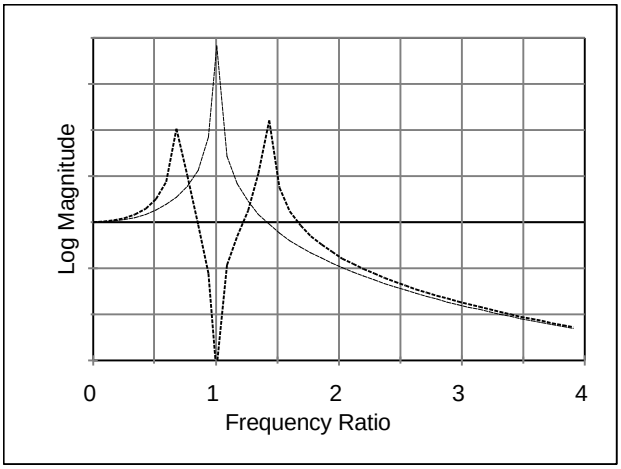
"There's no such thing as a free lunch!" When we modify a structure by adding an absorber of this type, we always create an extra natural frequency – we have converted a SDOF into a 2DOF. The two new natural frequencies are found by solving for $D = 0$. Non-dimensional quantities are also introduced to make the solution easier to interpret.

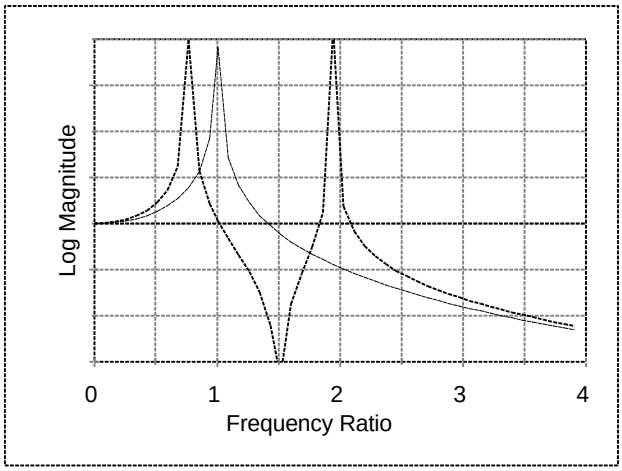
$$w_{1,2}^2 = \left(\frac{k_1}{2m_1}\right)\left(1 + q^2(1+m) \pm \sqrt{q^4(1+m)^2 - 2(1-m)q^2 + 1}\right)$$

with

$$q = \frac{\text{target frequency}}{\text{original natural frequency}} = \frac{\sqrt{k_2/m_2}}{\sqrt{k_1/m_1}} \quad \text{and} \quad m = \text{mass ratio} = \frac{m_2}{m_1}$$

The figures shown below compare the displacement X_1 before and after we add the absorber (both examples use a mass ratio of $\mu = 0.5$). In the first figure, the absorber target frequency was the original natural frequency ($q = 1.0$). In the second figure, the target frequency was $q = 1.5$. We are not including damping, so the heights of the peaks and troughs are not shown correctly (they should go to ±infinity on these graphs).





From the figures, note the following:

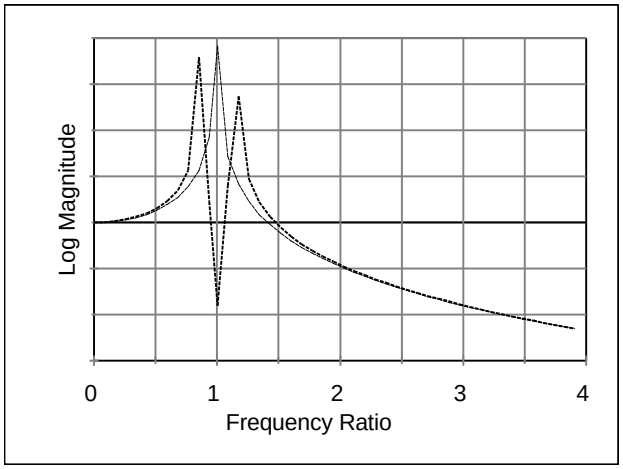
At the target frequency, the response of the primary mass has been reduced to zero (negative infinity on a log scale). This frequency is sometimes called an *anti-resonance.*

The new system has two natural frequencies with one natural frequency higher, and one lower than the original natural frequency of the SDOF -- the frequency at which the zero-amplitude response occurs is always between the two new natural frequencies.

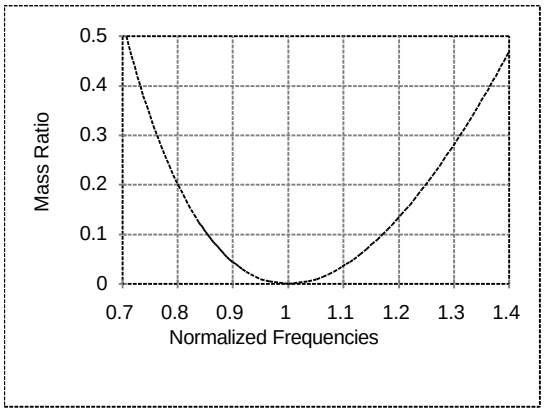
The very low frequency and very high frequency responses are essentially unchanged.

EFFECT OF THE SIZE OF THE SECONDARY MASS m_2

We also need to consider the difference between the two new natural frequencies, that is (w_2 - w_1). The problem is that if the excitation frequency changes a small amount (for example, an engine who's running speed changes slightly), and the two natural frequencies are close together, we may hit one of the new resonances of the modified system - we haven't achieved anything, and we have probably made the situation worse! It is not easy to see from the equations(!), but for a given target frequency, as the mass ratio increases there is a bigger difference between ω_1 and w_2, i.e., (w_2 - w_1) gets bigger. The next figure gives an example for a mass ratio of μ = 0.1 (this means we are only adding a small secondary mass) while targeting the original natural frequency (q = 1.0). Compare it to the previous figure for μ = 0.5 and note how reducing the secondary mass has brought the two peaks closer together



The figure to the right shows (for q = 1.0) how the new natural frequencies depend on the mass ratio. Select a mass ratio from the vertical axis, and read off the two new natural frequencies from the horizontal axis. The frequencies are normalized to the original natural frequency of the SDOF.



NOTE AND REMEMBER THAT $\sqrt{(k_2/m_2)}$ IS **NOT** A NATURAL FREQUENCY OF THE MODIFIED SYSTEM.

ABSORBER SUMMARY. A vibration absorber can be used to eliminate unwanted steady-state vibrations from a one-degree-of-freedom system. We do this by tuning the natural frequency of the absorber (by itself) to the excitation frequency. The following must be kept in mind when designing an absorber:

> The vibration of the primary mass is reduced to zero (for no damping), or very small (with light damping). However, this is ONLY at a single operating speed. If the system operates over a wide speed range, the vibrations away from the absorber frequency may still be large.
>
> At the absorber frequency, the vibration of the added secondary mass, m_2, depends on the *force* applied to the primary mass. The amount of vibration of the secondary mass may be very large compared to the original motion.
>
> The original system had one DOF. The modified system has an additional natural frequency. One of the new natural frequencies will be lower, and one higher, than the original natural frequency.
>
> The difference between the two new natural frequencies depends on the ratio of the secondary to primary masses, m_2/m_1 - the bigger the mass ratio, the further apart are the two new natural frequencies.

EXAMPLE A 1000 kg motor runs at a constant speed of 3600 rpm. Unfortunately, this is close to the natural frequency of the motor on its foundation. Design a SDOF vibration absorber to reduce the vibration of the motor.

We need to identify the combination of added mass and stiffness we wish to use, and also we need to consider some design specifics.

Target frequency: The running speed is constant, at 60 Hz. Therefore, the added mass/spring ratio must satisfy:

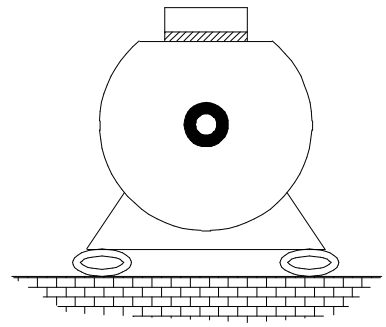
$$\frac{k_2}{m_2} = w^2 = \left(60 \times 2p\right)^2$$

Added mass: We need to keep both new resonances well away from the running speed. Normally, main electric supplies are maintained very close to 60 Hz, and therefore for this example we do not need to worry too much about the motor significantly changing speed. (Specifications for military ship borne generator speeds are somewhat more lenient, and so we would have to change the design for a ship borne application.) Let's assume we want the new resonances such that $w_1 < 55$ Hz and $w_2 > 65$ Hz. Looking at the previous figure (or calculating the numbers from the equations), this indicates the required mass ratio must satisfy $\mu = m_2/m_1 > 0.025$. Our added mass must therefore be at least 25 kg. This mass could be achieved with a block of steel with sides approximately 200 × 200 × 80 mm^3.

Required stiffness: The required stiffness is calculated from the target frequency and the (now decided) added mass:

$$k_2 = m_2 w^2 = (25)\times(60\times 2p)^2 = 3.55 \text{ MN/m}$$

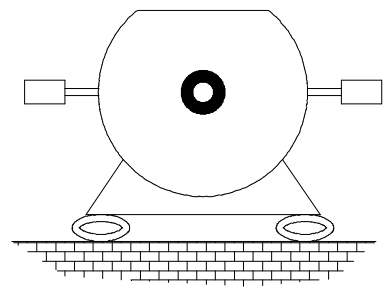
Design: We do not have any engineering drawings of the motor to help with a specific design. Therefore let us speculate that the top is relatively flat. This will allow us to stand the added mass on top of the motor, using a rubber pad for the stiffness element. In reality, the mass would be bolted, with rubber washers between the mass and bolts, and we would also take into account their stiffness.



Rubber pad: What size pad do we need? The pad we have available has a bulk modulus of k = 680 kN/m^2 (kPa). To convert this number to an actual spring stiffness, multiply by the contact area of the pad, and divide by its thickness. Lets choose the pad area to be 0.04 m^2 to match the size of the steel block. We can now calculate the thickness of the rubber pad:

$$\text{thickness} = \frac{680\times 10^3 \times 0.04}{3.55\times 10^6} = 7.7 \text{ mm}$$

Alternative design: What if we wish to split the secondary mass in two, and make the absorber something like the Stockbridge damper? We could do this by welding cantilevers onto each side of the motor casing, and fastening on the secondary masses. If our cantilevers are made from 25 × 13 mm^2 stock steel bar, how long should they be?



From *Strength of Materials,* when a cantilever is loaded at its tip with a force W (= m/g), the deflection is:

$$d = \frac{WL^3}{3EI}$$

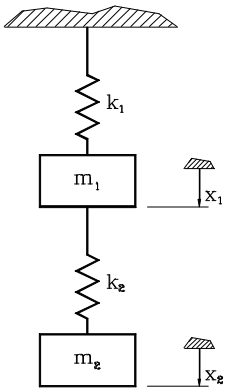
The second Moment of Area for our section is $I = (bt^3/12)$, and we need the stiffness = W/d = 1.775 MN/m (remember we are dividing the mass in two, so we only need half of the stiffness in each cantilever). The only unknown is the length of the cantilever.

$$\frac{W}{d} = \frac{3EI}{L^3} = \frac{3\times 210\times 10^9 \times \left(0.025\times 0.013^3/12\right)}{L^3} = 1.775\times 10^6 \text{ N/m}$$

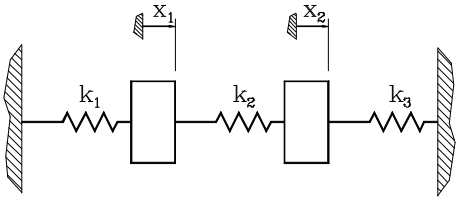
Solving: L = 117.6 mm.

ASSIGNMENTS

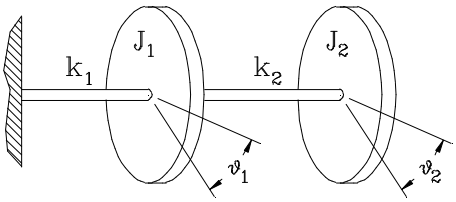
1. Given $m_1 = m_2 = m$ and $k_1 = k_2 = k$ for the system shown, write the equations of motion, and determine the two natural frequencies and mode shapes.



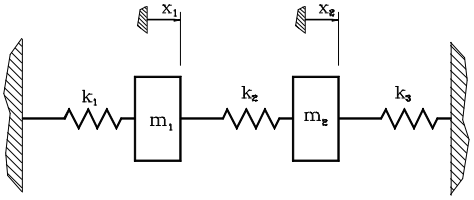
2. For the system shown, $m_1 = m_2 = m$. Determine the natural frequencies and mode shapes for the system when:
   a) $k_1 = k_2 = k_3 = k$
   b) $k_1 = k_2 = k$ and $k_3 = 2k$



3. Determine the natural frequencies and mode shapes for the torsional system shown, given the torsional spring rates are $k_1 = k_2$ and the mass moments of inertia are $J_1 = J_2$.



4. Determine the stiffness and flexibility matrices for the given system. Verify that the flexibility matrix is the inverse of the stiffness matrix.



5. A grinding wheel of 50 kg is firmly mounted on a table of 20 kg and rotates at 100 revolutions per minute. A misbalance in the wheel means that when it is running it produces a peak force of 15 N. Design a vibration absorber that fits under the tabletop. Assume there is a maximum permissible absorber deflection of 2.5 mm.

6. For the same system as problem 5, calculate the two natural frequencies of the modified system. Assume the table legs provide an equivalent stiffness of 2500 N/m.

7. For the same system, find the speed range through which the absorber reduces the vibration.

SOLUTIONS

1. Given $m_1 = m_2 = m$ and $k_1 = k_2 = k$ for the system shown, write the equations of motion, and determine the two natural frequencies and mode shapes.

First, find the natural frequencies:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Let

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} e^{iw_r t}$$

$$\begin{bmatrix} 2k - mw_r^2 & -k \\ -k & k - mw_r^2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$$

$$\left(2k - mw_r^2\right)\left(k - mw_r^2\right) - k^2 = 0$$

$$m^2 w_r^4 - 3kmw_r^2 + k^2 = 0$$

$$w_1^2 = 0.382\frac{k}{m} \;; \qquad w_2^2 = 2.618\frac{k}{m}$$

Now find the mode shapes. Expand the matrix equation (first line) to get:

$$\left(2k - mw_r^2\right)X_1 - kX_2 = 0$$

$$\frac{X_2}{X_1} = 2 - \frac{mw_r^2}{k}$$

$$\left(\frac{X_2}{X_1}\right)_1 = 1.618 \;; \qquad \left(\frac{X_2}{X_1}\right)_2 = -0.618$$

2. For the system shown, $m_1 = m_2 = m$. Determine the natural frequencies and mode shapes for the system when:

a) $k_1 = k_2 = k_3 = k$

$$\begin{vmatrix} \left(k_1 + k_2 - mw_r^2\right) & -k_2 \\ -k_2 & \left(k_2 + k_3 - mw_r^2\right) \end{vmatrix} = 0$$

$$\left(k_1 + k_2 - mw_r^2\right)\left(k_2 + k_3 - mw_r^2\right) - k_2^2 = 0$$

with $k_1 = k_2 = k_3 = k$

$$\left(2k - mw_r^2\right)\left(2k - mw_r^2\right) - k^2 = 0$$

$$w_1^2 = \frac{k}{m} \;; \qquad w_2^2 = \frac{3k}{m}$$

$$\left(\frac{X_2}{X_1}\right) = \frac{k_1 + k_2 - mw_r^2}{k_2}$$

$$\left(\frac{X_2}{X_1}\right)_1 = 1.00 \;; \qquad \left(\frac{X_2}{X_1}\right)_2 = -1.00$$

b) $k_1 = k_2 = k$ and $k_3 = 2k$

$$\left(2k - mw_r^2\right)\left(3k - mw_r^2\right) - k^2 = 0$$

$$w_1^2 = \frac{1.382k}{m} \;; \qquad w_2^2 = \frac{3.618k}{m}$$

$$\left(\frac{X_2}{X_1}\right) = \frac{k_1 + k_2 - mw_r^2}{k_2}$$

$$\left(\frac{X_2}{X_1}\right)_1 = 0.618 \;; \qquad \left(\frac{X_2}{X_1}\right)_2 = -1.618$$

3. Determine the natural frequencies and mode shapes for the torsional system shown, given the torsional spring rates are $k_1 = k_2$ and the mass moments of inertia are $J_1 = J_2$.

First find the natural frequencies:

$$\begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = 0$$

Let

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix} e^{iw_r t}$$

$$\begin{bmatrix} 2k - Jw_r^2 & -k \\ -k & k - Jw_r^2 \end{bmatrix}\begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix} = 0$$

$$\left(2k - Jw_r^2\right)\left(k - Jw_r^2\right) - k^2 = 0$$

$$J^2 w_r^4 - 3kJw_r^2 + k^2 = 0$$

$$w_1^2 = 0.382\frac{k}{J} \;; \qquad w_2^2 = 2.618\frac{k}{J}$$

Now the mode shapes. Expand a line of the matrix to get:

$$2k - Jw_r^2\Theta_1 - k\Theta_2 = 0$$

$$\frac{\Theta_2}{\Theta_1} = 2 - \frac{Jw_r^2}{k}$$

$$\left(\frac{\Theta_2}{\Theta_1}\right)_1 = 1.618\ ; \qquad \left(\frac{\Theta_2}{\Theta_1}\right)_2 = -0.618$$

4. Determine the stiffness and flexibility matrices for the given system. Verify that the flexibility matrix is the inverse of the stiffness matrix.

$$[M] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}; \qquad [K] = \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & (k_2 + k_3) \end{bmatrix}; \qquad [H] = \begin{bmatrix} \dfrac{(k_2 + k_3)}{\Delta} & \dfrac{k_2}{\Delta} \\ \dfrac{k_2}{\Delta} & \dfrac{(k_1 + k_2)}{\Delta} \end{bmatrix}$$

with $\Delta = k_1k_2 + k_1k_3 + k_2k_3$

verification: $[H][K] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

5. A grinding wheel of 50 kg is firmly mounted on a table of 20 kg and rotates at 100 revolutions per minute. A misbalance in the wheel means that when it is running it produces a peak force of 15 N. Design a vibration absorber that fits under the tabletop. Assume there is a maximum permissible absorber deflection of 2.5 mm.

From the handout (you should be able to derive this, so don't rely on remembering this equation) the motion of the secondary mass is given by:

$$X_2 = \frac{-F_1}{k_2}$$

Assuming we use the entire clearance for the motion of the absorber (this will also lead to the smallest secondary mass). We can calculate the absorber stiffness as:

$$k_2 = \frac{F}{x} = \left|\frac{15}{2.5\times10^{-3}}\right| = 6 \text{ kN/m}$$

We now need to determine the absorber mass. Recall that for the absorber to work, its natural frequency (before it is fastened to the vibrating system) is the same as the target (or excitation) frequency. Hence:

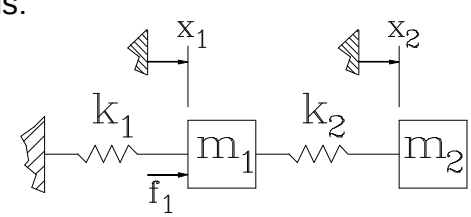
$$w^2 = \frac{k_2}{m_2}; \qquad m_2 = \frac{k_2}{w^2}$$

$$m_2 = \frac{6\times10^3}{\left\{(200/60)2p\right\}^2} = 13.68 \text{ kg}$$

This gives a mass ratio of m = 0.195, which is fairly small. This means that if mass is not critical for the design, we could increase the secondary mass. We would also have to change the secondary stiffness (increase it) as well.

6. For the same system as problem 5, calculate the two natural frequencies of the modified system. Assume the table legs provide an equivalent stiffness of 2500 N/m.

The total modified system is:



with m_1 = 70 kg; m_2 = 13.68 kg; k_1 = 2500 N/m; k_2 = 6000 N/m. From the 2DOF handout (again, don't remember this equation, work it out from first principles) the natural frequencies are the roots of:

$$\left(k_1 + k_2 - m_1 w_r^2\right)\left(k_2 - m_2 w_r^2\right) - k_2^2 = 0$$

$$\left(2500 + 6000 - 70 w_r^2\right)\left(6000 - 13.68 w_r^2\right) - 6000^2 = 0$$

$$957.6 w_r^4 - 536280 w_r^2 + 15 \times 10^6 = 0$$

$$w_1^2 = 29.53 \text{ rad}^2/\text{s}^2 \;; \quad w_2^2 = 530.5 \text{ rad}^2/\text{s}^2$$

$$w_1 = 5.43 \text{ rad/s} \;; \quad f_1 = 0.865 \text{ Hz}$$

$$w_2 = 23.0 \text{ rad/s} \;; \quad f_2 = 3.66 \text{ Hz}$$

7. For the same system, find the speed range through which the absorber reduces the vibration.

The literature uses several different ideas about solving this problem. Here, we take it to mean that the motion of the primary mass, X_1 is reduced by adding the secondary mass. Initially, the primary mass is a SDOF, and we can find its motion from:

$$X_1 = \frac{F_1}{m_1\left(w_n^2 - w^2\right)} = \frac{F_1}{\left(k_1 - m_1 w^2\right)}$$

For the 2DOF, the motion of the mass m_1 is found from (Guess what! You should work this from first principles – NOT an equation to remember!):

$$X_1 = \frac{F_1\left(k_2 - m_2 w^2\right)}{\left(k_1 + k_2 - m_1 w^2\right)\left(k_2 - m_2 w^2\right) - k_2^2}$$

Equating the two solutions finds the frequencies at which the SDOF and the 2DOF have the same amplitude. Strictly, here we are only interested the magnitude of the functions, so we have to reverse the sign of one of the equations before we equate them. Here's why - ask the prof to show you the graphs!

For a SDOF *below* resonance the force and displacement are in phase.

For a 2DOF *between the lower resonance and the vibration absorber frequency*, the force and displacement are out of phase.

So, for the lower solution we are looking for we have to introduce 180° (i.e. a sign change)

For a SDOF *above* resonance the force and displacement are out of phase.

For a 2DOF *between the vibration absorber frequency and the upper resonance*, the force and displacement are in phase.

So, for the upper solution we also have to introduce 180° (i.e. a sign change)

After some manipulation we get the following quadratic in w^2:

$$2m_1m_2w^4 - w^2\left(2k_1m_2 + 2k_2m_1 + k_2m_2\right) + 2k_1k_2 = 0$$

Substituting numbers and solving:

$$1915.2w^4 - 990480w^2 + 30\times10^6 = 0$$

$$w^2 = 32.3 \text{ or } 484.9 \text{ rad}^2/\text{s}^2$$

$$w_1 = 5.68 \text{ rad/s} ; \quad w_2 = 22.0 \text{ rad/s} ; \quad f_1 = 0.905 \text{ Hz} ; \quad f_2 = 3.50 \text{ Hz}$$