MECHANICAL ENGINEERING DEPARTMENT
UNITED STATES NAVAL ACADEMY

EM423 - INTRODUCTION TO MECHANICAL VIBRATIONS

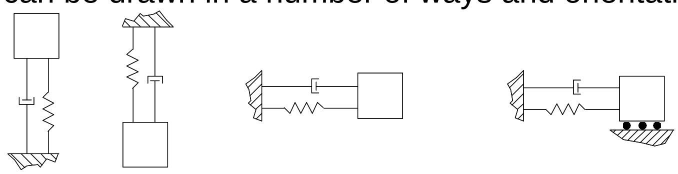
SINGLE DEGREE OF FREEDOM SYSTEMS
PART 1: UNFORCED MOTION

INTRODUCTION

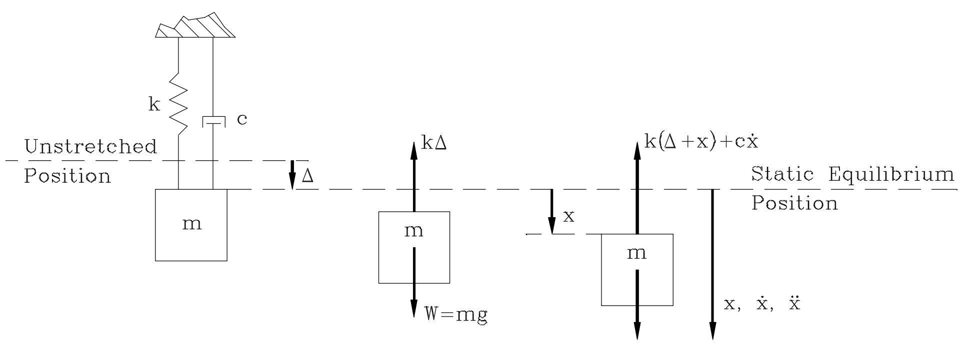
This section of the course covers a single degree of freedom mass, spring, damper system (SDOF). More complex systems are often analyzed by combining the effect of several SDOFs, and so a good understanding of the SDOF element is necessary.

THEORY

The SDOF element can be drawn in a number of ways and orientations:



For the moment, let us consider the system with the mass hanging down. The spring will stretch by an amount D, called the *static deflection*. We will be measuring the dynamic motion, x, from the static equilibrium position:



The FBD of the mass for the STATIC case reveals:

$$\sum \text{forces} = 0$$
$$mg - k\Delta = 0$$
$$mg = k\Delta$$

The FBD of the mass for the DYNAMIC case also includes the spring and damper terms:

$$\sum \text{forces} = m\ddot{x}$$
$$mg - k\left(\Delta + x\right) - c\dot{x} = m\ddot{x}$$

but we already know that

$$mg = k\Delta$$

therefore

$$\boxed{m\ddot{x} + c\dot{x} + kx = 0}$$

This equation is the basic equation of motion for the unforced response of a SDOF oscillator. Remember that the deflection is measured from the static equilibrium position. We often find that for systems that are stiffness controlled, providing the dynamic motion is measured from the static equilibrium position, the effects of gravity (i.e. its weight) do not appear in the equation.

UNDAMPED BEHAVIOR

For many mechanical systems the amount of damping (or energy dissipation) is 'small', so we first concentrate on the undamped SDOF. To do this we define

$$\boxed{\frac{k}{m} = w_n^2}$$

which makes the equation of motion

$$\boxed{\ddot{x} + w_n^2 x = 0}$$

This is satisfied by several solutions, but the one we use here is:

$$x(t) = A.\sin(w_n t) + B.\cos(w_n t)$$

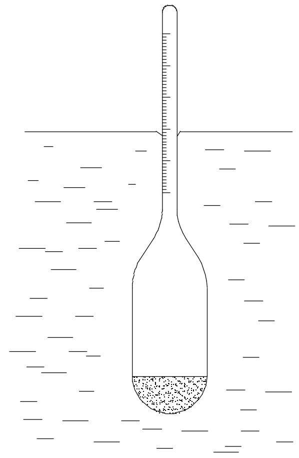
This solution shows the motion is harmonic with respect to time at frequency w_n (rad/s), which we call the “natural frequency.”

FINDING THE NATURAL FREQUENCY

There are several ways of finding the natural frequency of a SDOF. The following method determines the restoring force per unit displacement (or sometimes called the *equivalent stiffness*), and equates it to (mass) × (acceleration).

Example (This is a system that is NOT controlled by an actual spring stiffness, and we have to include the gravitational terms.)

A hydrometer float is used to measure the specific gravity of liquids. The hydrometer mass is 0.04 kg and the diameter of the cylindrical section protruding above the fluid is 0.006 m. Determine the period of vibration when the hydrometer bobs in a fluid with specific gravity 1.20.



$$\text{mass} = m = 0.04 \text{ kg}$$

$$\text{diameter} = d = 0.006 \text{ m}$$

$$\text{density} = r = 1.2 \times 10^3 \text{ kg/m}^3$$

We want the restoring force per unit "extension":

$$\text{buoyancy for extension } d = \text{ weight of displaced fluid}$$

$$= p\,\frac{(0.006)^2}{4}\left(1.2\times10^3\times9.81\right)d$$

When d is unity, this gives the "equivalent stiffness" or "*k*" term, so:

$$w_n^2 = \frac{k}{m} = \frac{p\,\frac{(0.006)^2}{4}\left(1.2\times10^3\times9.81\right)}{0.04} = 8.32 \text{ s}^{-2}$$

$$w_n = 2.885 \text{ rad/s} \qquad f_n = \frac{w_n}{2p} = 0.459 \text{ Hz} \qquad t_n = \frac{1}{f_n} = 2.18 \text{ s}$$

THE ENERGY METHOD

An alternative method of finding the natural frequency uses an energy approach. Overall this is often the more useful and powerful method. We consider the total kinetic energy in the inertias, and the potential (elastic) energy in the spring elements.

$$\text{Kinetic Energy } = T = \frac{1}{2} m\dot{x}^2 \qquad\qquad \text{Spring Energy } = U = \frac{1}{2} kx^2$$

Now remembering the basic solution of the equation of motion, we have

$$x(t) = A.\sin(w_n t)$$
$$\dot{x}(t) = A.w_n \cos(w_n t)$$

hence

$$T = \frac{1}{2} mA^2 w_n^2 \cos^2(w_n t) \qquad \mathrm{U} = \frac{1}{2} kA^2 \sin^2(w_n t)$$

from which

$$\mathrm{T}_{MAX} = \frac{1}{2} mA^2 w_n^2 \qquad U_{MAX} = \frac{1}{2} kA^2$$

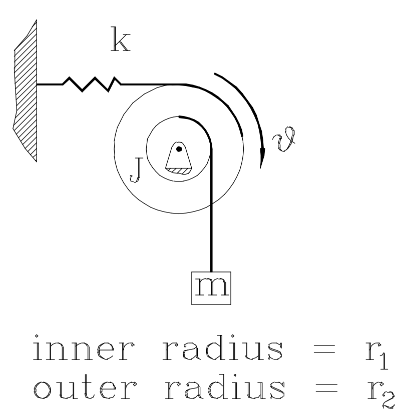
Comparing and remembering that $w_n^2 = \dfrac{k}{m}$ we get:

$$\mathrm{T}_{MAX} = U_{MAX}$$

In other words, the maximum kinetic energy is equal to the maximum potential energy.

Note that during a cycle of vibration the instantaneous amounts of kinetic and potential energy vary. The system has maximum kinetic energy at the instant that the potential energy is a minimum (zero). Conversely, at the instant when the potential energy is a maximum, the kinetic energy is zero. During a typical cycle, the SUM OF POTENTIAL AND KINETIC ENERGY IS CONSTANT, and the energy is transferred back and forth between the kinetic and potential forms. It is important to note that we are NOT saying that the KE = PE at any instant in time, but that the MAXIMUM values are the same. Also remember that this development assumes no energy dissipation.

Example Determine the natural frequency of the system:



We assume the motion is harmonic, with the angle of motion measured from the static equilibrium position. Both the pulley and mass have kinetic energy. Potential energy is stored in the spring. Remember that since we are measuring everything from the static equilibrium position, we do not include the gravitational potential energy term.

$$\text{Let } q\left(t\right) = A.\sin\left(w_n t\right)$$
$$\dot{q}\left(t\right) = A.w_n\cos\left(w_n t\right)$$

so

$$T_{MAX} = \left[\left(\frac{J\dot{q}^2}{2}\right)_{DISK} + \left(\frac{m\left(r_1\dot{q}\right)^2}{2}\right)_{MASS}\right]_{MAX} = \left(\frac{w_n^2 A^2}{2}\left(J + mr_1^2\right)\right)$$

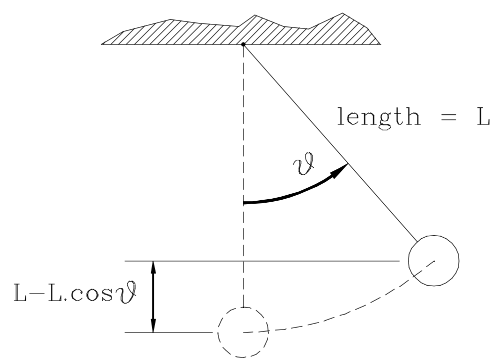
$$U_{MAX} = \left[\frac{k\left(r_2 q\right)^2}{2}\right]_{MAX} = \left(\frac{kr_2^2 A^2}{2}\right)$$

Equating T_{MAX} = U_{MAX} yields the natural frequency:

$$w_n = \sqrt{\frac{kr_2^2}{J + mr_1^2}} \text{ rad/s}$$

<u>Natural frequency of a simple pendulum</u>

The simple pendulum consists of a point mass at the end of a light string. This is another example of a system where the gravitiational potential energy has to be included in the calculations. The energy method is a good one for finding the natural frequency of this system.



$$U = \text{ Gravitational PE } = mgL\left(1 - \cos\left(q\right)\right)$$

$$T = \text{ Kinetic Energy, KE } = \frac{mv^2}{2} = \frac{m\left(L\dot{q}\right)^2}{2}$$

Let

$$q\left(t\right) = A.\sin\left(w_n t\right)$$
$$\dot{q}\left(t\right) = A.w_n\cos\left(w_n t\right)$$

We now apply the small angle approximations:

$$\sin\left(q\right) \approx q \qquad \text{and} \qquad \cos\left(q\right) \approx 1 - \frac{q^2}{2}$$

this gives

$$U_{MAX} = mgL\left(1-\left(1-\frac{A^2}{2}\right)\right) = \frac{mgLA^2}{2}$$

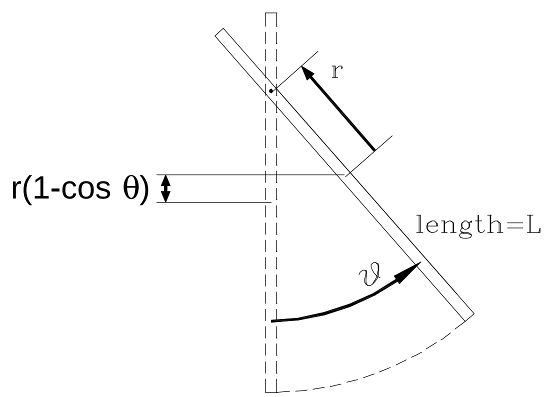
$$T_{MAX} = \frac{mL^2\omega_n^2A^2}{2}$$

Equating $T_{MAX} = U_{MAX}$ yields the natural frequency:

$$\omega_n = \sqrt{\frac{g}{L}}\ \text{rad/s} \qquad f_n = \frac{1}{2\pi}\sqrt{\frac{g}{L}}\ \text{Hz}$$

<u>Natural frequency of a compound pendulum</u>

The definition of a simple pendulum requires that the rotary inertia (energy) of the pendulum bob (mass) is small and can be ignored. A compound pendulum is one where the rotary inertia cannot be ignored. The example considered here is a uniform rod, length L, pivoted distance r from its center.



The kinetic energy includes both translation (of the center of gravity) and rotation:

$$T = \frac{1}{2}m\left(r\dot{\theta}\right)^2 + \frac{1}{2}I_G\dot{\theta}^2$$

$$\text{with} \qquad I_G = \frac{mL^2}{12} \qquad \text{and} \qquad \theta(t) = A.\sin\left(\omega_n t\right)$$

hence

$$T_{MAX} = \frac{1}{2}mr^2A^2\omega_n^2 + \frac{1}{2}\frac{mL^2A^2\omega_n^2}{12} = \frac{1}{2}m\omega_n^2A^2\left(r^2 + \frac{L^2}{12}\right)$$

The potential energy is similar to the potential energy term for a simple pendulum.

$$U = \text{ Gravitational PE } = mgr\left(1-\cos(\theta)\right)$$

Applying the same small angle approximations:

$$U_{MAX} = mgr\left(1-\left(1-\frac{A^2}{2}\right)\right) = \frac{1}{2}mgrA^2$$

Equating $T_{MAX} = U_{MAX}$ yields the natural frequency:

$$w_n = \sqrt{\frac{rg}{\left(r^2 + \frac{L^2}{12}\right)}} \text{ rad/s} \qquad f_n = \frac{1}{2p}\sqrt{\frac{rg}{\left(r^2 + \frac{L^2}{12}\right)}} \text{ Hz}$$

VISCOUS DAMPING

Viscous damping is a method of energy dissipation that is caused by a force that is of the form:

$$F_D = c\dot{x}$$

where c is a constant with units of force per unit speed. This type of damping is often considered in vibrations because it is reasonably representative of the energy dissipation methods in some engineering structures. It is also mathematically the easiest to apply to transient vibrations. Including both the energy loss through viscous damping and applying an excitation force modifies the equation of motion to:

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

The general solution to this second order differential equation is in two parts: the solution to the homogeneous equation (with $f(t) = 0$), and the particular integral (when $f(t)$ is non zero). The homogeneous equation represents a damped SDOF, with no external force excitation.

$$m\ddot{x} + c\dot{x} + kx = 0$$

Assuming a general solution of the form:

$$x(t) = Xe^{st}$$

gives:

$$\left(ms^2 + cs + k\right)Xe^{st} = 0$$

from which:

$$s_{1,2} = \frac{-c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

and so the general solution becomes:

$$x(t) = A.e^{s_1 t} + B.e^{s_2 t}$$

where A and B are constants that can be evaluated from the initial (time = zero) conditions. Substituting in the solutions for s_1 and s_2 gives:

$$x(t) = e^{-\left(\frac{c}{2m}\right)t}\left\{A.e^{+\left(\sqrt{\left(\frac{c}{2m}\right)^2-\frac{k}{m}}\right)t} + B.e^{-\left(\sqrt{\left(\frac{c}{2m}\right)^2-\frac{k}{m}}\right)t}\right\}$$

The first term represents a function that decays as a negative exponential with respect to time. However, the terms inside the { } depend on the values of the square-root terms. These terms may be real, zero or complex, depending on the relationships between k, c and m. When

$$\left(\frac{c}{2m}\right)^2 > \left(\frac{k}{m}\right)$$

the exponential terms in the general solution are all real, no oscillations are possible, and the SDOF is termed <u>over damped.</u> We do not cover this case in this course.

When:

$$\left(\frac{c}{2m}\right)^2 < \left(\frac{k}{m}\right)$$

the exponential terms are imaginary, and the SDOF can oscillate. This case is referred to as <u>under damped,</u> and is the case most often encountered in mechanical engineering vibrations. The limiting case is when

$$\left(\frac{c}{2m}\right)^2 = \left(\frac{k}{m}\right)$$

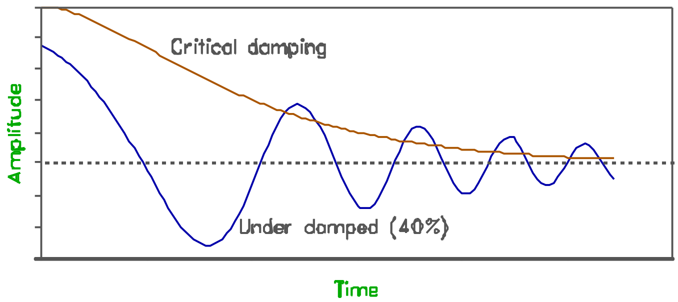
The limiting value for c is referred to as <u>critical damping,</u> and is given by:

$$\boxed{\text{critical damping } = c_c = 2m\sqrt{\frac{k}{m}} = 2mw_n = 2\sqrt{km}}$$

Note that the critical value of damping is a *theoretical* quantity that depends solely on the mass and stiffness of the system. IT DOES NOT TELL US ANYTHING ABOUT THE ACTUAL DAMPING of the system.

Actual damping is usually quoted relative to critical levels, and is referred to by the non dimensional term *viscous damping ratio.*

$$\boxed{\text{viscous damping ratio } = z = \frac{c}{c_c} \qquad (0 \le z \le 1)}$$



Using this relationship for viscous damping ratio, and rearranging the equation of motion, we get:

$$\ddot{x} + 2zw_n\dot{x} + w_n^2 x = \frac{f(t)}{m}$$

This non dimensional form of the equation is the one that is most useful.

The time solution to this problem has already been given and is restated:

$$x(t) = e^{-\left(\frac{c}{2m}\right)t}\left\{A.e^{+\left(\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right)t} + B.e^{-\left(\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right)t}\right\}$$

Remembering that

$$e^{\pm iwt} = \cos(wt) \pm i.\sin(wt)$$

we can rewrite the time solution as:

$$\begin{aligned} x(t) &= X.e^{-zw_n t}\sin(w_d t + j) \\ &= e^{-zw_n t}\left(C_1\sin(w_d t) + C_2\cos(w_d t)\right) \end{aligned}$$

with $\quad w_d = w_n\sqrt{1 - z^2}$

This introduces the *damped natural frequency*:

$$\boxed{\text{damped circular natural frequency (rad/s)} = w_d = w_n\sqrt{1 - z^2}}$$

LOGARITHMIC DECREMENT

One way of experimentally measuring the amount of damping in a system is to use the logarithmic decrement method. This method requires the system to be set in motion, then left to decay freely (i.e. without any more forcing). Using the following solution for the resulting "ring-down" motion:

$$x(t) = X.e^{-zw_n t}\sin(w_d t + j)$$

we consider the motion over N cycles of motion, starting at time $t = t_O$ and ending at time $t = t_N$. With N as an integer (i.e. we are considering a whole number of cycles), we find that

$$\sin(w_d t_N + j) = \sin(w_d t_O + j)$$

and

$$w_d\,(t_N - t_O) = 2Np$$

We look at the ratios of the amplitude of motion at these 2 times:

$$\frac{x_O}{x_N} = \frac{e^{-zw_n t_O}\sin(w_d t_O + j)}{e^{-zw_n t_N}\sin(w_d t_N + j)} = \frac{e^{-zw_n t_O}}{e^{-zw_n t_N}}$$

Take great care to note the difference between subscripts: n indicates natural frequency; N is the number of cycles. Let us define a *"logarithmic decrement"*, d_N.

$$d_N = \ln\left(\frac{x_O}{x_N}\right) = \ln\left(e^{-zw_n t_N - (-zw_n t_N)}\right)$$

$$= \ln\left(e^{zw_n (t_N - t_O)}\right)$$

hence $\quad d_N = zw_n\,(t_N - t_O)$

We further process this equation to eliminate the frequency term, since we do not know either the natural frequency, w_n or the damped natural frequency, w_d:

$$(t_N - t_O)w_d = 2Np$$

hence $\quad (t_N - t_O)w_n\sqrt{1 - z^2} = 2Np$

or $\quad w_n\,(t_N - t_O) = \dfrac{2Np}{\sqrt{1 - z^2}}$

This expression can be substituted into the equation for logarithmic decrement.

$$d_N = z w_n \left(t_N - t_O\right) = \frac{2Np}{\sqrt{1 - z^2}}$$

Finally, we make the simplifying assumption that damping is 'light', in which case $\sqrt{1 - z^2} \approx 1$, which reduces the logarithmic decrement equation to:

$$d_N = 2Npz \qquad \text{from which} \qquad z = \frac{d_N}{2Np}$$

<u>SUMMARY:</u>

First determine the logarithmic decrement, d_N, from measurements:

$$d_N = \ln\left(\frac{x_O}{x_N}\right)$$

For heavy damping, determine z from the quadratic:

$$d_N = \frac{2Npz}{\sqrt{1 - z^2}}$$

For light damping (z < 10 *%),* determine z from the equation:

$$z = \frac{d_N}{2Np}$$

<u>Logarithmic decrement example</u>
Consider the free decay shown below. Calculate the viscous damping ratio. As you work through the example, note the following:

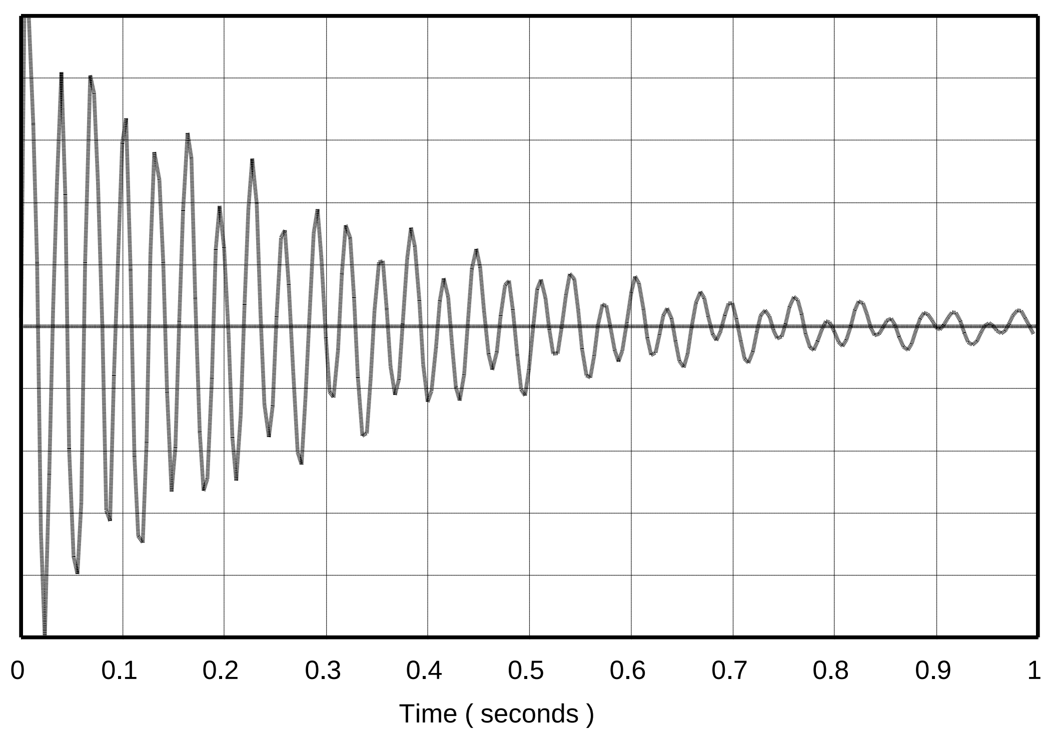
a) You do not need knowledge of the *absolute* motion. Relative (uncalibrated) motion is OK.

b) It does not matter whether the trace shows displacement, velocity or acceleration. Why?

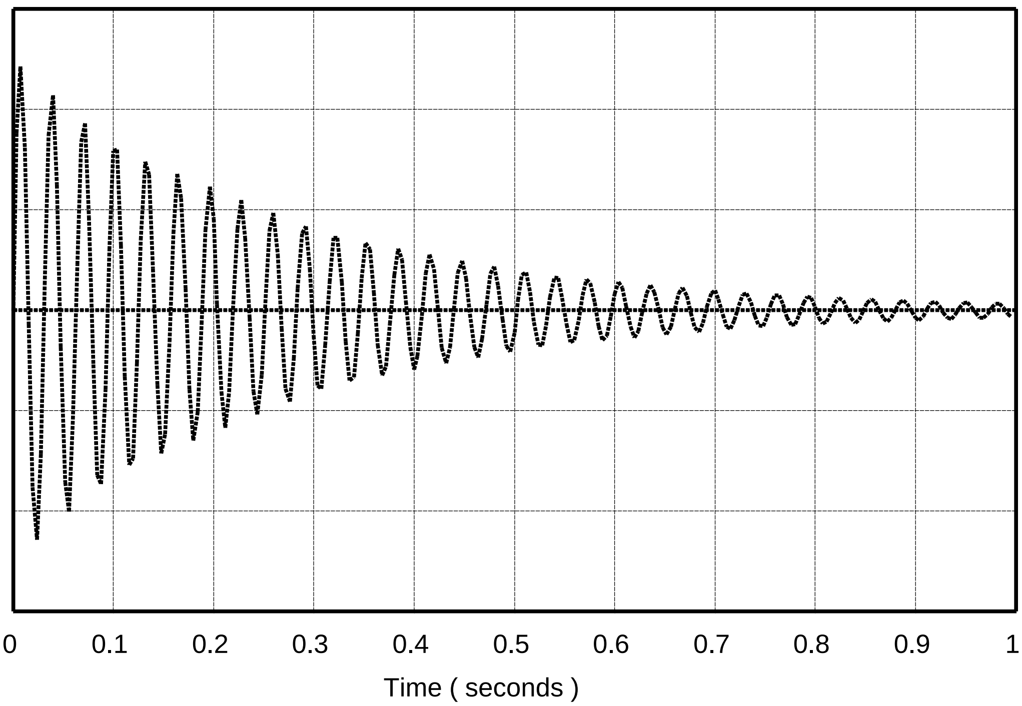
c) Why is the trace not a *pure* decaying sinusoid?
   i) Several modes

   ii) Digital aliasing

   iii) Signal noise



Now repeat the calculations for the data shown below. This is the same as the previous data, but with the *problems* removed.



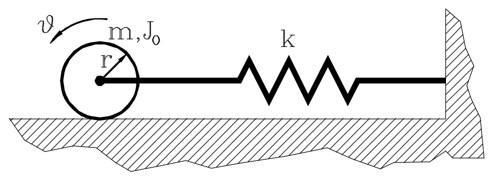
ASSIGNMENTS

1. A 0.4 kg mass is hung on a light spring, and causes it to elongate 8 mm from its unstretched length. Determine the natural frequency of the system.

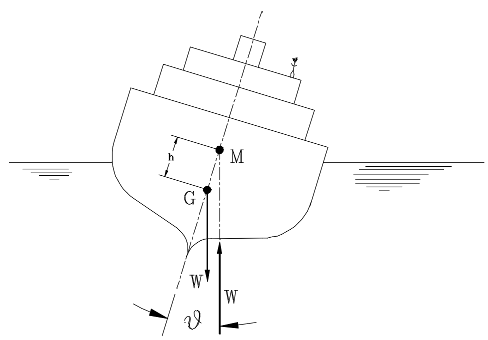
2. An unknown mass, *m*, is hung on a light spring, and causes it to elongate 3 mm from its unstretched length. Determine the natural frequency of the system.

3. An unknown mass, *m*, attached at the end of an unknown spring, *k*, has a natural frequency of 95 Hz. When a 0.5 kg mass is added to *m*, the natural frequency is lowered to 75 Hz. Determine the mass, *m* (kg), and the spring constant, *k* (N/m).

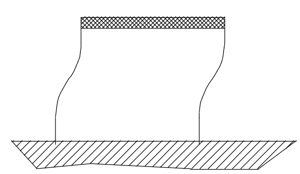
4. A cylinder of mass *m* and mass moment of inertia J_O rolls without slipping, but is restrained by the spring, *k*, as shown. Determine the natural frequency of the system.



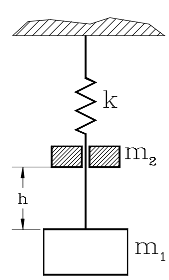
5. The oscillatory characteristics of ships in rolling motion depend on the position of the metacenter, *M*, with respect to the center of gravity, *G*. The metacenter, M, represents the point of intersection of the line of action of the buoyant force and the centerline of the ship, and its distance, *h*, measured from *G*, is the metacentric height, as shown. The position of *M* depends on the shape of the hull, and, for small values the angular inclination, *q*, of the ship, *h* is independent of *q*. Show that the period of motion for rolling is given by $t = 2p\sqrt{\frac{J}{Wh}}$ where *J* is the mass moment of inertia of the ship about its axis of roll, and *W* is the weight of the ship. Note: Unlike this problem, in general the position of the roll axis is unknown, and *J* is obtained from the period of oscillation measured from a test of a model.



6. The figure shows a simplified model of a single-story building. The columns are assumed to be rigidly embedded at the ends, and the deck has mass *m*. Determine the natural period, *t*. Note: The stiffness of a fixed-fixed spring is *(12 EI) /* L^3.



7. A mass m_1 hangs from a spring, *k* N/m, and is in static equilibrium. A second mass, m_2, is dropped from rest through height *h*, onto mass m_1. The masses stick together (no bounce). Determine the resulting oscillatory motion, referenced to the equilibrium position of the two masses hanging on the spring. *Hint: Use conservation of energy for the fall of* m_2*, then conservation of momentum for the impact. This gives the initial velocity condition for the resulting vibration.*



SOLUTIONS

1. A 0.4 kg mass is hung on a light spring, and causes it to elongate 8 mm from its unstretched length. Determine the natural frequency of the system.

$$K = \frac{w}{d} = \frac{0.4 \times 9.81}{8 \times 10^{-3}} = 490.5 \text{ N/m}$$
$$f_n = \frac{w_n}{2p} = \frac{1}{2p}\sqrt{\frac{k}{m}} = \frac{1}{2p}\sqrt{\frac{490.5}{0.4}} = 5.57 \text{ Hz}$$

2. An unknown mass, m, is hung on a light spring, and causes it to elongate 3 mm from its unstretched length. Determine the natural frequency of the system.

$$K = \frac{w}{d} = \frac{mg}{3 \times 10^{-3}} = 3270m \text{ N/m}$$
$$f_n = \frac{w_n}{2p} = \frac{1}{2p}\sqrt{\frac{k}{m}} = \frac{1}{2p}\sqrt{\frac{3270m}{m}} = 9.10 \text{ Hz}$$

3. An unknown mass, m, attached at the end of an unknown spring, k, has a natural frequency of 95 Hz. When a 0.5 kg mass is added to m, the natural frequency is lowered to 75 Hz. Determine the mass, m (kg), and the spring constant, k (N/m).

$$f_1 = \frac{1}{2p}\sqrt{\frac{k}{m}} = 95 \text{ Hz} \quad \text{hence} \quad k = (95 \times 2p)^2 m$$
$$f_2 = \frac{1}{2p}\sqrt{\frac{k}{(m+0.5)}} = 75 \text{ Hz} \quad \text{hence} \quad k = (75 \times 2p)^2 (m+0.5)$$

Simultaneous solution yields:

m = 0.8272 kg ; k = 294.7 kN/m

4. A cylinder of mass m and mass moment of inertia J_O rolls without slipping, but is restrained by the spring, k, as shown. Determine the natural frequency of the system.

$$q = A.\sin(w_n t) \qquad x = rA.\sin(w_n t)$$
$$\dot{q} = Aw_n \cos(w_n t) \quad \dot{x} = rAw_n \cos(w_n t)$$

$$T_{MAX} = \left\{ \frac{1}{2} m\dot{x}^2 + \frac{1}{2} J_O \dot{q}^2 \right\}_{MAX} = \frac{1}{2}\left(mr^2 + J_O\right) w_n^2 A^2$$

$$U_{MAX} = \left\{ \frac{1}{2} kx^2 \right\}_{MAX} = \frac{1}{2} kr^2 A^2$$

$$\text{Equate } T_{MAX} = U_{MAX} : \qquad \left(mr^2 + J_O\right) w_n^2 = kr^2$$

$$w_n^2 = \frac{kr^2}{\left(mr^2 + J_O\right)} \ ; \qquad f_n = \frac{w_n}{2p} = \frac{1}{2p}\sqrt{\frac{k}{\left(m + \frac{J_O}{r^2}\right)}}$$

5. The oscillatory characteristics of ships in rolling motion depend on the position of the metacenter, *M*, with respect to the center of gravity, *G*. The metacenter, M, represents the point of intersection of the line of action of the buoyant force and the centerline of the ship, and its distance, *h*, measured from *G*, is the metacentric height, as shown. The position of *M* depends on the shape of the hull, and, for small values the angular inclination, *q*, of the ship, *h* is independent of *q*. Show that the period of motion for rolling is given by $t = 2p\sqrt{\frac{J}{Wh}}$ where *J* is the mass moment of inertia of the ship about its axis of roll, and *W* is the weight of the ship. Note: Unlike this problem, in general the position of the roll axis is unknown, and *J* is obtained from the period of oscillation measured from a test of a model.

Moments about G:

$$-Whq = J\ddot{q}$$

Let

$$q(t) = A.\sin(w_n t) \ ; \qquad \ddot{q} = -w_n^2 A.\sin(w_n t)$$

Hence

$$w_n = \sqrt{\frac{Wh}{J}} \ ; \qquad t_n = \frac{1}{f_n} = \frac{2p}{w_n} = 2p\sqrt{\frac{J}{Wh}}$$

6. The figure shows a simplified model of a single-story building. The columns are assumed to be rigidly embedded at the ends, and the deck has mass *m*. Determine the natural period, t. Note: The stiffness of a fixed-fixed spring is *(12 EI) / L^3*.

$$k = 2\times\left(\frac{12EI}{L^3}\right) ; \qquad t_n = \frac{2p}{w_n} = 2p\sqrt{\frac{m}{k}} = 2p\sqrt{\frac{mL^3}{24EI}}$$

7. A mass m_1 hangs from a spring, *k* N/m, and is in static equilibrium. A second mass, m_2, is dropped from rest through height *h*, onto mass m_1. The masses stick together (no bounce). Determine the resulting oscillatory motion, referenced to the equilibrium position of the two masses hanging on the spring. *Hint: Use conservation*

of energy for the fall of m_2, then conservation of momentum for the impact. This gives the initial velocity condition for the resulting vibration.

$$\text{Speed of } m_2 \text{ just before impact} = V_O = \sqrt{2gh}$$
$$m_2 V_O = \left(m_1 + m_2\right)\dot{x}\left(0\right)$$
$$\text{so} \quad \dot{x}\left(0\right) = \frac{m_2\sqrt{2gh}}{\left(m_1 + m_2\right)}$$

The initial displacement is (negative) the increase in static equilibrium displacement.

$$x\left(0\right) = \frac{-\left(m_1 g + m_2 g\right)}{k} + \frac{m_1 g}{k} = \frac{-m_2 g}{k}$$

General solution:

$$x = A.\cos\left(w_n t\right) + B.\cos\left(w_n t\right) \quad \text{with} \quad w_n^2 = \frac{k}{\left(m_1 + m_2\right)}$$

Apply the initial conditions:

$$A = \frac{-m_2 g}{k} \qquad B w_n = \frac{m_2\sqrt{2gh}}{\left(m_1 + m_2\right)} \quad \text{hence} \quad B = \frac{m_2\sqrt{2gh}}{\sqrt{k\left(m_1 + m_2\right)}}$$

Therefore

$$x\left(t\right) = \left(\frac{m_2 g}{k}\right)\cos\left(w_n t\right) + \left(\frac{m_2\sqrt{2gh}}{\sqrt{k\left(m_1 + m_2\right)}}\right)\sin\left(w_n t\right) \quad \text{with} \quad w_n = \sqrt{\frac{k}{\left(m_1 + m_2\right)}}$$