

## LETTERS AND COMMENTS

## Hanging angles of two electrostatically repelling pith balls of different masses

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### Abstract

An analytic solution can be derived for the angles of two mutually repelling charged pith balls of unequal mass hanging from strings from a common point of attachment. Just as in the equal-mass case, a cubic equation is found for the square of the sine of either angle, and an approximation can be used to avoid Cardano's formula for small angles. These results extend a standard problem treated in introductory undergraduate courses in electricity and magnetism.

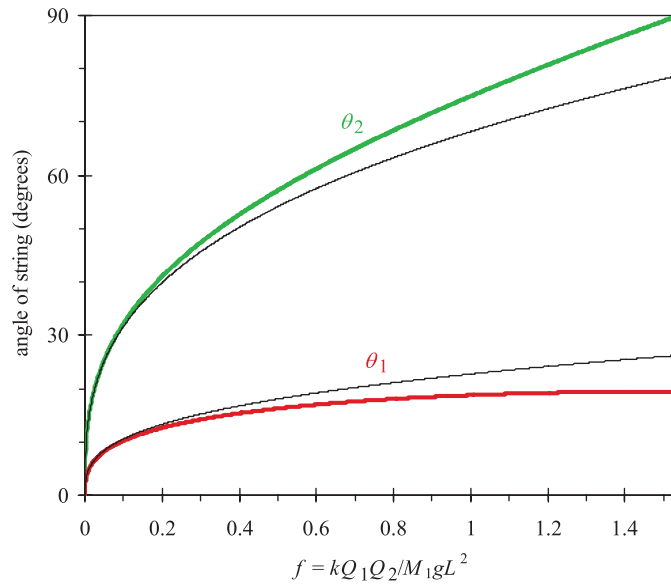
A recent paper [1] investigated two positively charged pith balls hanging from a nail at the end of strings of equal length  $L$ . The problem consisted in finding their hanging angles,  $\theta_1$  and  $\theta_2$ , when the balls do not have the same masses ( $M_1$  and  $M_2$ ) or charges ( $Q_1$  and  $Q_2$ ). A numerical solution was developed. In this letter, it is shown that an analytic solution also exists using Cardano's equation, which deserves occasional exercise by physics students.

Equation (9) in [1] can be squared, the terms proportional to  $C$  isolated, and that expression can be squared again to eliminate the remaining square roots. The resulting equation appears to be a sixth-order polynomial in  $s^2$ , but it actually reduces to a cubic one. One nice way to make that fact plausible without resorting to an algebraic software package is to substitute in various values for  $m \equiv M_1/M_2$  and  $f \equiv kQ_1Q_2/M_1gL^2$  to discover that the coefficients of the zeroth, fifth and sixth powers are always zero. Motivated by that goal, one can obtain after a fair bit of algebra

$$a_3x^3 + a_2x^2 + a_1x + a_0 = 0, \quad (1)$$

where  $x \equiv \sin^2 \theta_1$ . The coefficients of this cubic equation are

$$\begin{aligned} a_3 &= 4(m+1)^4/f^2, \\ a_2 &= m(m-1)^2, \\ a_1 &= 4m + 3(m-1)^2 \quad \text{and} \\ a_0 &= f^2(m-1)^2/16 - 4. \end{aligned} \quad (2)$$



**Figure 1.** Hanging angles when the first pith ball is three times as massive as the second ball. The thick coloured curves plot the exact results by solving equation (1) using Cardano's formula, while the thin dark curves are the approximate results from equation (4).  
(This figure is in colour only in the electronic version)

As a check, equations (1) and (2) correctly reduce to equation (10) in [1] for  $m = 1$ , the equal-mass case. But note that for  $m > 1$  none of the coefficients are negative for  $f = f_{\max} \equiv 8/(m-1)$ , implying that the only possible solution of  $x$  (which is the square of a sine and thus cannot be negative) is zero, and thus  $\theta_1 = 0$  and  $\theta_2 = \pi$ . This result clearly explains why the balls hang vertically (with ball 1 directly below the nail and ball 2 directly above it) whenever  $f \geq f_{\max}$ , which was a surprising outcome of the numerical computations in [1]. Such explanations demonstrate an important advantage of the analytic solution presented here.

Cardano's formula can be used to find the unique real root  $x$  of equation (1). Then  $\theta_1 = \sin^{-1} \sqrt{x}$  whose solution in the first quadrant is adopted. Finally,  $\theta_2 = \sin^{-1}(m\sqrt{x})$  where the first-quadrant solution of the inverse sine is adopted if  $f < f_c$  and the second-quadrant solution if  $f > f_c$ , where  $f_c$  is the critical angle defined by equation (12) of [1].

The two hanging angles can therefore be analytically calculated as a function of  $f$  for any given value of  $m$ . For example, consider the case  $m = 3$  for which  $f_c = 8\sqrt{3}/9 \approx 1.54$  and  $f_{\max} = 4$ . The exact solutions for the two hanging angles  $\theta_1$  and  $\theta_2$  are plotted in figure 1 for  $f$  ranging from 0 up to  $f_c$  (at which point  $\theta_2 = 90^\circ$  and  $\theta_1$  has its maximum value).

When the two pith balls have equal mass, introductory textbooks [2] restrict  $f$  to be low enough in value that one can invoke the small-angle approximation to obtain

$$\theta_1 = \theta_2 \approx \left(\frac{f}{4}\right)^{1/3} \quad (3)$$

(in radians) and thereby avoid use of Cardano's formula. This result follows immediately from equation (10) of [1] by dropping the small term  $f^2x$ . By similarly assuming that both  $f \ll 1$

and  $x \ll 1$ , equations (1) and (2) can be approximated to the lowest order as

$$\theta_1 \approx \left[ \frac{f}{(m+1)^2} \right]^{1/3} \quad \text{and} \quad \theta_2 \approx m \theta_1 \quad (4)$$

for the unequal-mass case, which by inspection reduces to equation (3) when  $m = 1$ . Equation (4) is plotted for  $m = 3$  in figure 1. These approximate curves are in good agreement with the exact results near the origin, but increasingly diverge from them as  $f$  gets large. Equation (4) can be quickly deduced from equation (7) of [1], using the standard lowest-order approximations for cosine and sine. Deriving equation (4) would make a good challenge problem for undergraduate physics majors in an introductory electricity and magnetism course.

## References

- [1] Mungan C E 2011 Electrostatic repulsion of charged pith balls hanging from strings *Eur. J. Phys.* **32** 207–12
- [2] Young H D and Freedman R A 2008 *University Physics* 12th edn (San Francisco, CA: Addison-Wesley) p 746