The following appendix is taken directly from our recent submission to Physics of Fluids, where Mcilhany, Guth and Wiggins define three dimensional and higher dimensions Eulerian Indicators [2]. The three dimensional definition of transversality is given in the main body of the text. Here, the other EIs are defined in three dimensions, by first defining them in higher dimensions (N) and the applying the definitions to N = 3.

**Appendix: Eulerian Indicators in N-Dimensions**

The three dimensional character of the Chesapeake Bay requires that all EIs be defined in three dimensions. In most cases, changing the former definition from two to three dimensions was trivial as the original definition was set in Cartesian coordinates, allowing an easy extension from two to three dimensions simply by adding a “z” component to each vector used in the calculation. Two of EIs however, required more care in extending them to three dimensions, namely, transversality, α, and transverse shear, β. Transverse shear required a redefinition as originally, it was defined by finding the gradient along the line perpendicular to the flows direction. In moving to three dimensions, the line becomes a plane and then the issue is in deciding what gradient to take within the plane normal to the flow direction. Transversality proved to be more difficult to translate into three dimensions as it is a measure of the spread of angles from the mean flow direction. In redefining the calculation to three dimensions, the issue becomes which angle should be used to consider how transverse the set of vectors is to itself?

This appendix will show that each EI can be defined not only in three dimensions, but be extended into N-dimensions as well. Most of the appendix will center on the redefinition of transversality. Extending the definitions of EI from two to three dimensions was a necessary change, however, extending to N dimensions does not seem justifiable from the perspective of fluid flow, given that we live in a three dimensional world. Other data sets however exist that exhibit dynamics and can live in higher dimensional spaces; for example, a state vector for a stock index such as the S&P500, the pixel information from a hyper-spectral image (over many planes), an insurance database. N-dimensional flows can exist, the question is whether these data sets have physical properties that would benefit from the application of either Lagrangian analysis or EIs.

### Transverse Shear, RROC and Mobility in N-Dimensions

A quick review of the definitions given in this paper for transverse shear, β, the relative rate of change, RROC, and the mobility, η, show that in each case, the definitions are applicable from two dimensions up to an arbitrarily higher number of dimensions, N. In the case of the transverse shear, the gradient operator acts on the N-dimensional velocity vector magnitude, giving the N-dimensional gradient vector, ρ. The transverse shear, given by Eqn. 3, is defined as the transverse component of ρ relative to \( \hat{v} \).

\[
\rho(r, t) = \nabla |v(r, t)|
\]

\[
\rho(r, t) \cdot \hat{v} = |\rho(r, t)| \cos(\phi)
\]

\[
\beta(r, t) = |\rho(r, t)| \sin(\phi)
\]

where \( \phi \) is the angle between \( \rho \) and \( \hat{v} \).

### Transverse Shear

The transverse shear, denoted as \( \beta \), measures how much a fluid packet experiences a shear in the perpendicular direction relative to the flows overall direction. Figure 1 illustrates how this can shearing can occur. When a location in a flow has high transverse shear, fluid particles tend to drift away from their current trajectories onto new ones. At a given point in the velocity field, the gradient of the magnitude of that velocity vector (shown as \( \rho \)) is projected onto the velocity vector. From this, the transverse component is easily found, dubbed \( \beta \).
Although an obvious change would be to redefine $\beta$ as the magnitude of the cross product of $\rho$ and $v$, the best reason to keep the definition as it is presented in equations 6 is that these definitions are trivially extendable into higher dimensions.

\begin{align*}
\rho(r, t; \mu) &= \nabla|v(r, t; \mu)| \\
\rho(r, t; \mu) \cdot \hat{v} &= |\rho(r, t; \mu)| \cos(\phi) \\
\beta(r, t; \mu) &= |\rho(r, t; \mu)| \sin(\phi)
\end{align*}

where $\phi$ is the angle between $\rho$ and $\hat{v}$.

\[ RROC \]

Transverse shear differs from the other EIs in that it solely depends on its spatial structure, i.e., it is time independent. All other EIs in this study depend on a collection of velocity vectors over time. In the case of transversality, the collection serves to define both the mean and the spread of the distribution. The RROC requires several points in a time-series of velocity vectors to define an average rate of change and the mobility is an integral over time.

The Relative Rate of Change ($RROC$) has been defined previously [1], so a short summary of its properties is given in this section. The $RROC$ measures a scale independent rate of change of a velocity vector over time. The RROCs is sensitive to both changes in magnitude as well as direction, with instantaneous values peaking at a maximum of 1.0 whenever the velocity changes sign (in 1D) or rotates by 180° in one time step. The $RROC$ integrates the differential change, $\Delta w(r, t; \mu)$, which is bounded over the interval [0, 1] and defined next. The differential change is integrated over a time window given by the interval $[0, t_f]$, and then divided by that length time, yielding a rate. The $RROC$ is interpreted as the rate of change the velocity field undergoes to be used in contrast to the rate of change that particles in the fluid experience. When the value for the $RROC$ is low, the velocity field is near or at steady flow. When the value is high, the velocity field
is changing rapidly, possibly due to turbulence or when a location in the field is near a moving stagnation point.

\[
\Delta w(r, t; \mu) = \frac{\| v(r, t + \Delta t; \mu) - v(r, t; \mu) \|}{\| v(r, t + \Delta t; \mu) \| + \| v(r, t; \mu) \|} = \begin{cases} 
0 & \text{if } v(r, t + \Delta t; \mu) = v(r, t; \mu) \\
1 & \text{if } v(r, t + \Delta t; \mu) = -v(r, t; \mu) \\
1 & \text{if } v(r, t + \Delta t; \mu) = 0 \\
1 & \text{if } v(r, t; \mu) = 0 
\end{cases} 
\]  
(7)

\[
dw(r, t; \mu) = \lim_{\Delta t \to 0} \Delta w(r, t, \Delta t; \mu) 
\]  
(8)

\[
\frac{\partial w(r, t; \mu)}{\partial t} = \lim_{\Delta t \to 0} \frac{\Delta w(r, t, \Delta t; \mu)}{\Delta t} 
\]  
(9)

\[
RROC(r; \mu) = \bar{w}(r; \mu) = \frac{1}{t_f} \int_0^{t_f} \frac{\partial w(r, t; \mu)}{\partial t} dt, 
\]  
(10)

**Mobility**

The EI for transport dubbed mobility, \( \eta(r, t) \) measures the consistency of the velocity field to transport particles away from the location under consideration. Defined initially in a previous study [1], three basic components are required for this calculation, the \( RROC \), a characteristic time, \( T_c \), and the velocity integrals \( v_{fast} \) and \( v_{slow} \). The \( RROC \) is defined in Sec. . The characteristic time can be defined in a number of ways depending on the nature of the flow considered. For the Hill’s vortex model in this study, a characteristic time, \( T_c = 3.3 \), was used as this is the average recirculation time for particles of fluid in the unperturbed vortex case. The main purpose of the characteristic time is to give meaning to the rate found by the \( RROC \). The combination of the two values then properly weights the velocity integrals used in calculating the mobility.

The calculations for the velocity integrals are given below. When a flow is changing its direction quickly compared to the rate of passage of fluid particles, the velocity vectors are integrated first, then the magnitude for the total is dubbed \( v_{fast} \). When a flow changes direction too quickly, the net displacement from a position in the domain is small as the integral of the vector leads to a smaller overall displacement. When a velocity field is slowly changing its direction, or is steady, compared to the rate of passage of fluid particles, the overall displacement away from the location in the domain should be higher as the velocity vector is consistently pushing the fluid away from its position. In this case, the magnitude is taken first for the velocity vector and then integrated and dubbed \( v_{slow} \).

\[
v_{fast}(r; \mu) = \left| \int_0^{t_f} v(r, t; \mu) dt \right| 
\]  
(11)

\[
v_{slow}(r; \mu) = \int_0^{t_f} |v(r, t; \mu)| dt. 
\]  
(12)

The definition of “fast” and “slow” depends upon the time scale under consideration, and they are given meaning through the characteristic time scale, \( T_c \), and the \( RROC \), as follows:

\[
\eta'(r; t; \mu) \equiv \psi_{sum}(r; \mu) \equiv \left(1 - e^{-\bar{w}(r, \mu)T_c}\right) v_{fast}(r; \mu) + \left(e^{-\bar{w}(r, \mu)T_c}\right) v_{slow}(r; \mu) 
\]  
(13)

The mobility indicates the strength of transport away from its location in the domain. Stagnation points should have zero mobility, as should boundary locations where the flow approaches zero. Turbulent regions should also indicate frustrated transport over the time window of integration. Steady flow regions should
indicate high transport, unless the magnitude of the flow is low. As transport is necessary for any kind of effect in the flow to possible, the EI for transport, mobility, is used as a product to the other EIs. Transversality and Transverse Shear are additive as they represent two different properties of the flow, each to be enhanced by transport.

Extending the definitions (Eqn. 10 and 13) and interpretations for both the RROC and mobility, η, are trivial extensions using all N components of the velocity vector. The interpretations do not change as a result, where the RROC is still the relative rate of change of a vector and the mobility measures how well a position in the domain transports fluid particles away from that location.

In each of these EIs, extending from two dimensions to N-dimensions was trivial. The initial goal was simply to redefine the EIs in three dimensions, however, once it became clear that the EIs could be defined in N dimensions, the temptation to fully extend the set of definitions pushed for a redefinition of transversality as well, which proved to be more difficult.

Transversality in N-Dimensions

Transversality began as a simple measure of the change in angle for a flow defined by a discrete map either in time or spatially periodic (duct flow). At each transition in the mapping of the flow, the velocity vector at a point may undergo a change in direction such that change in angle of the velocity suggests “streamline-crossing” behavior [3], which has been noted as a condition which can promote mixing of the fluid. In changing from discrete maps to continuous changes in the flow, transversality was redefined as the spread of angles [1] around a mean angle over a window in time or space, depending on how the flow changes. When the definition of transversality is limited to two dimensions, the angle used to measure the deviation from the mean is the polar angle defined where the mean direction is the polar axis. In extending this definition from two to three to N-dimensions, additional angles are introduced into the problem, calling into question what additional transverse structure is created by the additional degrees of freedom.

Imagine a set of velocity vectors almost aligned. The mean vector defines the polar axis for the set. The standard deviation of the angles from the mean angle is approximately the transversality. Transversality is only concerned with how much a vector is perpendicular to the mean, as a result, the value for the standard deviation of the angles is folded to lie between [0°,90°]. By changing from two to three to N dimensions, each additional dimension adds a new angular degree of freedom. For example, a set of vectors lying uniformly distributed on the 30° cone from the polar axis has a different transversality than a set of vectors at 30° forming a cross distribution in the plane normal to the flow’s direction. In this case, there is an additional degree of transversality taking place among the angles lying in the normal plane. By this reasoning, a recursive formula was determined which captures the idea of transversality in N-dimensions.

Singular Value Decomposition (SVD)

The N-dimensional calculation of transversality begins by first performing a Singular Value Decomposition (SVD) of the velocity vector set formed where the tail of each vector is the origin and by taking all velocity vectors at a point in the domain from a window in time, \( t_{i-X} < t_i < t_{i+X} \), where X represents the number of samples in time both before and after the current time. SVD simultaneously calculates the scaling factors in N-dimensions as well as the orientation of the N-dimensional ellipsoid formed by the velocity vectors. The scaling factors measure the effective size of each dimension that the vector set spans. Using the scaling factors, a parameter is formed which measures the dimension “fullness” of the velocity vector set. The recursive relationship will depend only on the polar angle given from the highest dimensional axis and the next angle given by the next largest dimensional axis. The recursive relation then suggests the form the dimension fullness parameter should take dubbed, \( D_N \):

\[
D_N = \frac{c^2}{a \times b}
\]

where \( a, b \) and \( c \) are the first three scaling parameters given by the SVD, equivalent to the semi-major and semi-minor axes for a N-dimensional ellipsoid and \( a \geq b \geq c \). The first ratio \( \frac{c}{b} \) gives the oblateness value -
how close the distribution resembles a sphere or a clock face; where the second ratio, $\frac{c}{a}$ gives the scale factor that is needed to measure how collapsed the vector set is in a lower dimension. Extending this definition to the $N$th dimension:

$$D_N = \frac{a_{N-2}^2}{a_N \times a_{N-1}}$$

(15)

where $N$ represents the current dimension that the vector set has been reduced to; the fractions are interpreted in a similar way as before. It is worth noting that the dimension fullness parameter $D_N$, is an EI itself, being a product of the velocity field only.

\[\text{FIG. 2. Cases one (solid) and two (dashed) used in the normalization for the N-dimensional transversality calculation.}\]

In three dimensions, the unit vectors in case one are at $\theta_{ij} = 90^\circ$, in case two the three unit vectors lie on the $45^\circ$ cone, making the vectors equidistant at an angle of $\theta_{ij} = 75.5^\circ$.

**Transversality Recursive Relation**

The recursive formula defining $\alpha_N$ is:

$$\alpha_N = \alpha_{\theta,N} + D_N (p_N \alpha_{\theta,N} + q_{N,M} \alpha_{N-1})$$

(16)

where $\alpha_{\theta,N}$ is the transversality for the polar angles defined for the $N$th dimension and $p_N, q_{N,M}$ are admixture coefficients which are set by two normalizing cases. The highest dimension that the vector set spans is denoted by $M$ and is identified by SVD as the “largest” dimension. Finally, $\alpha_N$ and $\alpha_{N-1}$ are the transversality in the $N, N-1$ dimensions respectively. The form of this relationship implies that the lowest dimensions must be calculated first, then successively used to calculate the next highest dimension until $N = M$.

There are two values labeled as “alpha” used in this calculation, $\alpha_{\theta,N}$ and simply $\alpha_N$. The first value, $\alpha_{\theta,N}$ is the deviation of the polar angles from the mean angle, which is set along the polar axis, $\langle \theta \rangle = 0^\circ$. The second value, $\alpha_N$ is the transversality for the $N$th dimension, which takes into account any additional contributions to transversality based on how the vectors are distributed in the lower dimensions.

The two coefficients, $p_N, q_{N,M}$, can be found by considering two cases used for normalization. In case one, begin with an $N$-dimensional coordinate axes set in the positive quadrant, then rotate the set of unit vectors such that the mean vector becomes a new polar axis. Each unit vector will be at the same polar angle in the rotated system at $\theta \geq 45^\circ$. For three dimensions, the polar angle of the three unit vectors is $\theta = 54.7^\circ$.  

\[\text{(a)Cases one and two shown overlapping.} \]

\[\text{(b)Cases one and two rotated to show their relative angles.}\]
A key feature of this set of vectors is that they have preserved their angles such that all unit vectors are still \( \theta_{ij} = 90^\circ \) with respect to each other.

Case two considers a similar vector set, rotated such that the mean vector is a new polar axis. In this case, however, the rotated unit vectors are set at \( \theta = 45^\circ \). The angle between each of the unit vectors is now less than \( 90^\circ \), as they lie closer to each other, in other words, they are no longer orthogonal. For three dimensions, the angle between each vector is \( \theta_{ij} = 75.5^\circ \). Figure 2 illustrates the polar angles for the two test cases in three dimensions. One further advantage that these two test cases have is that the order of transitions between vectors does not affect the answer for the final transversality. In case one, the final transversality is \( \alpha_1^N = 90^\circ \) as each of the unit vectors is exactly \( 90^\circ \) with respect to each other, so that any sequence of transitions between the vectors always results in the final transversality being \( 90^\circ \). Similarly, for case two, the unit vectors are equidistant from each other separated by an angle, \( \theta_{ij} \). By a similar argument, any sequence of transitions from one unit vector to the other results in the same overall transversality, \( \alpha_2^N \), which is less then \( 90^\circ \), where superscripts have denoted the cases.

By choosing these two cases for normalization, several values can be established that will always be constant given a particular N-dimensional case. For case one, \( \alpha_1^N \) is maximal at \( 90^\circ \). For case two, the polar \( \alpha_{2,N} \) is maximal at \( 90^\circ \). By choosing these two cases, the mixing coefficients can be found, \([p_N, q_{N,M}]\). In both cases, the calculation for the highest dimension differs between the two, however, by “dropping” down one dimension, accomplished by removing the highest dimension coordinate from the vector, both cases become the same shape, an N-hedral simplex.

Figures 3(a) and 3(b) illustrate the idea of a coordinate system being reduced in dimension. By taking the unit vectors defined above and simply removing the highest dimensional coordinate, a new set of \( N \) vectors is created, defined over \( N - 1 \) dimensions. The structure created by lowering the dimension of a coordinate system is a set of vectors that are equally distant from each other in the lower dimension. This is best considered starting from three dimensions, then dropping down to two. A coordinate set of unit vectors, equally spaced from the polar axis in three dimensions will project into two dimensions as a “peace sign”, meaning three vectors spaced apart by \( 120^\circ \). By this same argument, a four dimensional coordinate systems unit vectors will project into three dimensions as four vectors equally spaced apart, forming a structure similar to a methane molecule \((CH_4)\), shown in figure 3(b). The angle separating each vector is \( 109.5^\circ \), known as the dihedral angle commonly used in chemistry. Higher dimensional N-hedral angles are defined as: \( \cos(\theta_H^N) = -\frac{1}{N} \).

For the cases described here, a notation is used where the case is indicated by a superscript and the
variable dependence is indicated by a subscript, \( \alpha_i \), for instance. Several angles are needed to define the values given in Eqn. 17, namely, the polar angle formed by each case (\( \theta_1 \), \( \theta_2 \), \( \theta_H \)), the angle between any of the vectors (\( \theta_{ij,N} \), \( \theta_{ij,N}^2 \), \( \theta_{ij,N}^H \)) the polar transversality (\( \alpha_{\theta,N}^1, \alpha_{\theta,N}^2, \alpha_{\theta,N}^H \)) and the transversality (\( \alpha_{\theta,N}^H, \alpha_{\theta,N}^2, \alpha_{\theta,N}^H \)). The operator, \( \hat{\mathcal{F}}_1 \), folds angles from \([0°...180°] \) about \( 90° \) and \( \hat{\mathcal{F}}_2 \) folds angles from \([0°...90°] \) about \( 45° \). The angles, dimension parameters, and transversality are defined as:

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2</th>
<th>( N )-hedral Simplex</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_N^1 = 1 )</td>
<td>( D_N^2 = \frac{1}{\sqrt{N-1}} )</td>
<td>( D_N^H = 1 )</td>
</tr>
<tr>
<td>( \theta_N^1 = \cos^{-1}\left(\frac{1}{\sqrt{N}}\right) )</td>
<td>( \theta_N^2 = 45° )</td>
<td>( \theta_N^H = \cos^{-1}\left(-\frac{1}{N}\right) )</td>
</tr>
<tr>
<td>( \theta_{ij,N}^1 = 90° )</td>
<td>( \theta_{ij,N}^2 = \cos^{-1}\left(\frac{1}{2} \frac{N-2}{N-1}\right) )</td>
<td>( \theta_{ij,N}^H = \theta_N^H )</td>
</tr>
<tr>
<td>( \alpha_{\theta,N}^1 = 180° - 2\theta_N^1 )</td>
<td>( \alpha_{\theta,N}^2 = 90° )</td>
<td>( \alpha_{\theta,N}^H = \hat{\mathcal{F}}_2 \left[ \sqrt{\frac{N}{M}} \left(180° - \theta_N^H\right) \right] )</td>
</tr>
<tr>
<td>( \alpha_N^1 = 90° )</td>
<td>( \alpha_N^2 = \theta_{ij,N}^2 = 75.5° )</td>
<td></td>
</tr>
</tbody>
</table>

Both normalization cases become the same structure, the \( N \)-hedral simplex after dropping down from the top dimension, \( M \). This allows the calculation to proceed identically between the two cases. Care must be taken when considering how the \( N \)-hedral simplex reduces in dimension. In each case, the previous higher dimensional polar axis is reduced to zero. The calculation of transversality computes the mean variance of angles from the new polar axis. The vectors which have been reduced to zero are still present in the calculation, however, they only contribute by keeping the total number of vectors at \( M \) and not \( N \), the current number of dimensions. For this reason, \( \alpha_{\theta,N}^H \) which leads to the second coefficient, \( q_{N,M} \) also depends on both \( M \) and \( N \). Although not proven in the above equations, the value for the dimension parameter, \( D_N \), is identically one for case one and the \( N \)-hedral simplex and \( \frac{1}{\sqrt{N-1}} \) for case two.

Starting from Eqn. 17 and writing out the two cases, solving for the coefficients is straight forward.

\[
\begin{align*}
\alpha_N^1 &= \alpha_{\theta,N}^1 + D_N^1 \left(p_N \alpha_{\theta,N}^1 + q_{N,M} \alpha_{\theta,N}^H\right) \\
\alpha_N^2 &= \alpha_{\theta,N}^2 + D_N^2 \left(p_N \alpha_{\theta,N}^2 + q_{N,M} \alpha_{\theta,N}^H\right) \\
\alpha_{N,M}^H &= \alpha_{\theta,N}^H + D_N^H \left(p_N \alpha_{\theta,N}^H + q_{N,M} \alpha_{\theta,N}^H\right) \\
p_N &= \sqrt{N-1} \left[\frac{90° - \theta_{ij,N}^2}{90° - 2\theta_N^1}\right] - 1 \\
q_{N,M} &= \frac{1}{\alpha_{\theta,N}^H} \left[90° - \alpha_{\theta,N}^1 (p_N + 1)\right]
\end{align*}
\]

By definition, the coefficients, \( p_1, p_2, q_{1,M}, q_{2,M} \equiv 0 \) as there is no transversality in one dimension and only the polar transversality in two dimensions. Starting from \( N = 2, M = 3 \), the values of \( \alpha_{N,M}^H \) are calculated,
each building the term needed by the next higher dimension:

\[ \alpha_{1,M} = 0 \]  
\[ \alpha_{2,M} = \alpha_{\theta,2,M} \]  
\[ \alpha_{3,M} = (1 + D_3^H p_3) \alpha_{\theta,3} + D_3^H q_3,M \alpha_{2,M} \]  
\[ \alpha_{4,M} = (1 + D_4^H p_4) \alpha_{\theta,4} + D_4^H q_4,M \alpha_{3,M} \]  
\[ \vdots \]  
\[ \alpha_{N,M} = (1 + D_N^H p_N) \alpha_{\theta,N} + D_N^H q_{N,M} \alpha_{N-1,M} \]  

The definitions of \( D_N^H \) and \( q_{N,M} \) reduce the formula for the \( N \) hedral simplex to:

\[ \alpha_{N,M}^H = \hat{F}_1 \left[ 90^\circ + (1 + p_N) \left( \alpha_{\theta,N,M}^H - \alpha_{\theta,N}^1 \right) \right] \]  

Using the formulae presented, calculated values for both \( p_N \) and \( q_{N,M} \) are given in table Tab. .

<table>
<thead>
<tr>
<th>( N )</th>
<th>( M )</th>
<th>( q_{N,M} )</th>
<th>( p_N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>2.00</td>
<td>1.93</td>
<td>2.16</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>2.05</td>
<td>1.75</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>2.10</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

TABLE I. Calculated values for the coefficients \( q_{N,M} \) and \( p_N \) needed for the evaluation of transversality, \( \alpha_N \). In the table, \( M \) represents the largest dimension the vector set spans and \( N \) represents the dimension for which the calculation is performed. The coefficients, \( q_{N,M} \) and \( p_N \), are admixture constants based on the normalization cases one and two.

**Calculation of Transversality in N-Dimensions**

The following steps describe how to calculate the \( N \)-dimensional transversality.

1. Collect a set of \( n \)-dimensional vectors \((v)\), where \( N_v \) denotes the size of the vector set and \( N \) denotes the number of dimensions that define the vectors.
2. Normalize the vector to the unit sphere.
3. Calculate the Singular Value Decomposition (SVD) of the vector set.
4. Rotate the vector set by the rotation matrix obtained from SVD to obtain a new set of vectors, \( v' = U^T v \), whose components are given by \( v'_i \), where \( i = 1...N \).
5. From the scaling matrix obtained from SVD, compute the set of dimension parameters, \( D_N = \frac{a_N^2}{a_{N-2} \times a_{N-1}} \).
6. Calculate the polar angle transversality, \( \alpha_{\theta,N} \)

   - Define the polar angle in the \( N^{th} \) dimension for the \( i^{th} \) vector: \( \theta_i = \cos^{-1} \left( \frac{v'_i}{\sqrt{\sum_{j=1}^{N} v'_j^2}} \right) \).
• Fold the polar angles to be less than 90°: \( \tilde{\theta}_i = \tilde{F}_1[\theta_i] \)

• Calculate the variance of the folded angle: \( \mathcal{V}(\tilde{\theta}_i) = \frac{1}{N_v} \sum_{i=1}^{N_v} (\tilde{\theta}_i)^2 \).

• Calculate the standard deviation of the variance: \( \sigma(\tilde{\theta}) = \sqrt{\mathcal{V}(\tilde{\theta})} \).

• Fold the rms value about 45° and normalize to 90° to obtain the polar transversality: \( \alpha_{\theta,N} = 2 \tilde{F}_2[\sigma(\tilde{\theta})] \).

• The whole calculation can be summarized in one formula: \( \alpha_{\theta,N} = 2 \tilde{F}_2\left[ \sqrt{\frac{1}{N_v} \sum_{i=1}^{N_v} \left( \tilde{F}_1[\theta_i] \right)^2} \right] \).

7. From the rotated set of vectors, \( \mathbf{v}' \), create subsets of vectors by removing the highest dimension’s coordinate, creating a set of \( N_v \) vectors for each lower dimension, \( \tilde{\mathbf{v}}'_{n} \), where \( n = 1...N - 1 \).

8. Using the recursive formula: \( \alpha_{N} = (1 + D_{N}p_{N}) \alpha_{\theta,N} + D_{N}q_{N,M} \alpha_{N-1} \), begin calculating the transversality from the lowest dimensions upwards until reaching the highest dimension, \( N \). In each of the lower dimensions, the set of vectors used to calculate \( \alpha_{\theta,N} \) is based on the respective set of lower dimensional vectors, \( \tilde{\mathbf{v}}'_{N} \).

\[
\begin{align*}
\alpha_1 &= 0 \\
\alpha_2 &= \alpha_{\theta,2} \\
\alpha_3 &= (1 + D_{3}p_{3}) \alpha_{\theta,3} + D_{3}q_{3,M} \alpha_2 \\
\alpha_4 &= (1 + D_{4}p_{4}) \alpha_{\theta,4} + D_{4}q_{4,M} \alpha_3 \\
&\quad \vdots \\
\alpha_N &= (1 + D_{N}p_{N}) \alpha_{\theta,N} + D_{N}q_{N,M} \alpha_{N-1}
\end{align*}
\]

