FM Noise and Performance


Note: This generally corresponds to Madhow, with some slight deviations. Madhow takes a more intuitive approach, Z&T use a more mathematical analysis.

Noise and Performance of FM Signals

The following is a block diagram of a basic FM demodulator:

Consider a generic FM signal:

\[ y_p(t) = A_c \cos \left( 2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau \right) \]

Note that the FM signal can be rewritten as:

\[ y_p(t) = A_c \cos \left( 2\pi f_c t + \phi(t) \right) \]

Which can be expressed as a time-varying phasor:

\[ \overline{Y}_p = A_c \angle \phi(t) \]

Generically, the output of the discriminator is (we’ll look at this in greater detail later):

\[ \hat{m}(t) = k_f m(t) + \frac{n'(t)}{2\pi A_c} \]

Signal Power: \[ P_s = k_f^2 P_m \]
On a phasor diagram, this looks like:

Recall: The phasor actually rotates in time (it would trace out a sinusoid if you looked at it edgewise). In this case, if we watched the phasor rotate, we would see the phase angle advance and delay over time in response to the FM modulation.

Now: Note that noise is essentially a random phasor that can be expanded into real and imaginary components (known as the inphase and quadrature components). For our analysis, we’re only concerned with noise at the carrier frequency (or instantaneous frequency), thus we can write our noise term as:

\[
n(t) = n_i(t) \cos(2\pi f_c t) - n_Q(t) \sin(2\pi f_c t)
\]

\[
\tilde{N} = \sqrt{n_i^2(t) + n_Q^2(t)} \tan^{-1}\left(\frac{n_Q(t)}{n_i(t)}\right)
\]

Note: Consider why if you ask the question “How do you demodulate FM” you get the answer “Take the ArcTangent of Q over I”.

Noise illustrated on a Phasor Diagram:

\[ \text{Note: } \phi_e(t) \text{ is the phase deviation error caused by the noise.} \]

What makes this complicated is that I don’t care about the amplitude variations (recall that we’ll remove those with a limiter). The only issue is the change in the phase angle, \( \phi_e(t) \), which is what introduces noise into our demodulated signal.

Through some math (see Ziemer and Tranter), we can write the input to the discriminator in the following manner:

\[ y_{BPF}(t) = R(t) \cos(2\pi f_c t + \phi(t) + \phi_e(t)) \]

and it can be shown that (see Ziemer and Tranter):

\[ \phi_e(t) = \tan^{-1}\left( \frac{n(t) \sin[\phi_n(t) - \phi(t)]}{A_e + n(t) \cos[\phi_n(t) - \phi(t)]} \right) \]

Now, what we’re concerned with:

\( \phi(t) \) conveys the message signal.
\( \phi_n(t) \) is the random phase angle associated with the noise.
\( \phi_e(t) \) is the noise component that adds to the message signal.
What’s going to be input to the discriminator is thus:

\[ y_{BPF}(t) = R(t)\cos\left(2\pi f_c t + \phi(t) + \phi_e(t)\right) \]
\[ y_{BPF}(t) = R(t)\cos\left(2\pi f_c t + \psi(t)\right) \]
\[ \psi(t) = \phi(t) + \phi_e(t) \]

If we assume that we’re operating in a high SNR case, then we can make the simplification that:

\[ A \gg n_r(t), \quad A \gg n_Q(t) \]

And, after some math, what falls out is:

\[ y_{BPF}(t) = R(t)\cos\left(2\pi f_c t + \psi(t)\right) \]
\[ \psi(t) = \phi(t) + \frac{n_Q(t)}{A_c} \]

**Note:** Preceding the discriminator with a limiter will remove the amplitude variations caused by \( R(t) \).

The output of the discriminator will then be:

\[ z(t) = \left(2\pi f_c + \frac{d\psi(t)}{dt}\right)\sin\left(2\pi f_c t + \psi(t)\right) \]

Now, recall that for an FM signal, we can write:

\[ \phi(t) = 2\pi k_f \int_0^t m(\tau) d\tau \]

With some substitutions, the output of the low-pass filter can be shown to be:

\[ \hat{m}(t) = 2\pi k_f m(t) + \frac{1}{2\pi A_c} \frac{dn_Q(t)}{dt} \]

For the sake of convenience, Madhow chooses to normalize out the \( 2\pi \) term on the signal, and is left with:

\[ \hat{m}(t) = k_f m(t) + \frac{1}{2\pi A_c} \frac{dn_Q(t)}{dt} \]
**Note:** Taking the derivative is the same as passing a function through a filter with response \( j2\pi f \).

If we look at the above equation, it can be shown that the Power Spectral Density of the noise component is:

\[
S_{\text{noise}}(f) = \left| j2\pi f \right|^2 \left( \frac{1}{2\pi A_c} \right)^2 (2) (S_n(f)) = \left( \frac{(j2\pi f)^2}{(2\pi A_c)^2} \right) (2) \left( \frac{N_0}{2} \right) = \frac{N_0 f^2}{A_c^2}
\]

Why the factor of 2: Because we have to account for positive and negative frequencies.

What this tells us is that we’ve shaped the spectrum of the noise (i.e., it’s no longer AWGN). Finding the SNR is no longer a trivial operation at this point. When plotted, the PSD of the noise now looks like:

![PSD Plot](image)

Notice that our operations have had a profound effect on the performance of FM systems operating in the presence of noise. It is clear that low frequency signals are subjected to much lower noise levels than high-frequency signals.

The total noise power is then the area under the curve:

\[
P_{\text{noise}} = \frac{2}{3} \frac{N_0}{A_c^2} f_m^3 = \frac{2}{3} \frac{N_0}{A_c^2} B_m^3
\]

**Note:** \( f_m \) is the maximum frequency of our original message signal.

The total signal power is then:

\[
P_s = k_f^2 P_m
\]
Thus, the SNR is:

\[ \text{SNR}_{\text{out}} = \frac{P_{\text{sig}}}{P_{\text{noise}}} = \frac{P_s = k_f^2 P_m}{\frac{2}{3} \frac{N_0}{A^2 f_m^3}} \]

For Sinusoidal Modulation: \( \text{SNR}_{\text{out}} = \frac{3 A_k^2 k^2}{4 N_0 f_m^3} \)

\[ \text{SNR}_{\text{out}} = \frac{3 A_k^2 k^2 P_m}{2 N_0 f_m^3} = \frac{3 A_k^2 k^2 P_m}{2 N_0 B_m^3} \]

**Note That:** We’ve traded off bandwidth inefficiency for improved performance (higher SNR).

**Note That:** For a sinusoid only, we can simplify the above equation:

\[ \text{SNR}_{\text{out}} = \frac{3 \beta^2 A_k^2}{4 N_0 f_m^3} \]
Lecture: FM Performance Example (Handout)

Given: An FM signal with amplitude of 10V, \( f_c = 144 \text{ MHz} \), \( \beta = 2.5 \), \( N_0 = 10^{-4} \text{ W/Hz} \). The modulating signal is a 1 kHz tone with an amplitude of 2V.

Find: (a) The bandwidth of the signal.
     (b) The SNR at the output of the Receiver.
     (c) Evaluate a method for improving the SNR at the output of the receiver.

(a) Given that \( \beta > 1 \), we can use the Carson’s Rule Bandwidth:

\[
BW \approx 2f_m (1 + \beta) \\
BW = 2(1\text{kHz})(1 + 2.5) \\
BW = 7\text{kHz}
\]

(b) First, we need to determine \( p_{sig} \)

Since the modulating signal is sinusoidal, this is simple:

\[
\overline{m(t)^2} = \frac{m_p^2}{2} = \frac{2^2}{2} = 2\text{Watts}
\]

We also need the deviation constant, \( k_f \):

\[
\beta = \frac{k_f m_p}{f_m} \Rightarrow k_f = \frac{\beta f_m m_p}{2V} = \frac{(2.5)(1\text{kHz})}{2V} = 1250\text{ Hz/V}
\]

Then, the SNR is just:

\[
\text{snr} = \frac{3A^2k_f^2\overline{m(t)^2}}{2N_0 f_m^3} = \frac{3(10)^2(1250\text{ Hz/V})^2(2W)}{2(10^{-4} \text{ W/Hz})(1\text{kHz})^3} = 4687.5
\]

\[
\text{SNR} = 36.7 \text{ dB}
\]
(c) Because we have a pure tone signal, we could utilize a bandpass filter centered around $f_m$ at the output of the detector. Note that the noise is quadratic, and the total noise power is simply the area under the curve.

Assume for the moment that we use a BPF with a center frequency of 1.0 kHz and a bandwidth of 100 Hz.

The new noise power is thus:

$$p_N = \int_{-\frac{BW}{2}}^{\frac{BW}{2}} \frac{4\pi^2 N_0 f^2}{A^2} df = \frac{8\pi^2 N_0 f^3}{3A^2} \bigg|_{-\frac{BW}{2}}^{\frac{BW}{2}} = 7901W$$

Thus, the SNR is:

$$SNR = \frac{4\pi^2 k^2 m(t)^2}{p_N} = \frac{4\pi^2 (1250 \text{ Hz} / V)^2 (2W)}{7901W} = 15614$$

$$SNR = 42.0 \text{ dB}$$

**Note That:** By bandlimiting the noise, we obtained an SNR improvement of 5.3 dB.