Learning Objectives

a. Define a phasor and use phasors to represent sinusoidal voltages and currents
b. Determine when a sinusoidal waveform leads or lags another Graph a phasor diagram that illustrates phase relationships
c. Define and graph complex numbers in rectangular and polar form
d. Perform addition, subtraction, multiplication and division using complex numbers and illustrate them using graphical methods
e. Represent a sinusoidal voltage or current as a complex number in polar and rectangular form
f. Define time domain and phasor (frequency) domain
g. Use the phasor domain to add/subtract AC voltages and currents
h. For purely resistive, inductive and capacitive elements define the voltage and current phase differences
i. Define inductive reactance
j. Understand the variation of inductive reactance as a function of frequency
k. Define capacitive reactance
l. Understand the variation of capacitive reactance as a function of frequency
m. Define impedance
n. Graph impedances of purely resistive, inductive and capacitive elements as a function of phase

Suppose you had to shape a block of steel into the USNA logo. One option would be to hammer and grind and bend and chisel the steel into shape—basically to use brute force.

Another option would be to heat the steel to a temperature at which it becomes soft. Then we can easily shape the steel into the desired form. Of course we must then cool the steel back to room temperature. So, in this second approach, we might say we transform the steel into another domain where it is easier to work with, do the necessary work (which is much easier since the steel is soft), then transform the steel back to its familiar form (by cooling it).

Stretching the analogy, we see that we invest work in transforming the steel and transforming it back, but in its transformed state it is much easier to work with (so that the effort in transforming and transforming back is worth it).

This general idea is useful in solving many engineering problems. We transform our problem into “another domain” where the problem is much easier to work with, solve the problem in the other domain, and then transform back to the “real world.” You may have heard of several of these transform techniques: the Laplace Transform, the Fourier Transform, the Hilbert Transform, the Z Transform, etc.

Solving AC circuit problems is greatly simplified through the use of the phasor transform. In fact, investing in this transform makes solving AC circuits no more difficult than solving DC circuits. But to understand phasors, we have to understand complex numbers… so let’s review that first.

Complex numbers As we discuss complex numbers, do not waste time contemplating the philosophy of complex numbers—complex numbers are merely an invention designed to allow us to talk about the quantity $j = \sqrt{-1}$ —nothing more. Your only concern should be visualizing complex numbers on a plot, and manipulating complex numbers.

A complex number is a number of the form $C = a + jb$ where $a$ and $b$ are real and $j = \sqrt{-1}$. $a$ is the real part of $C$ and $b$ is the imaginary part. Note that the letter $j$ is used in electrical engineering to represent the imaginary component since the letter $i$ already has heavy use as the symbol for current ($i$).
Geometric Representation  We represent complex numbers geometrically in two different forms.

In the rectangular form, the $x$-axis serves as the *real* axis and the $y$-axis serves as the *imaginary* axis. So, for example, the complex number $C = 6 + j8$ can be plotted in rectangular form as:

$$C = 6 + j8$$

(rectangular form)

**Example:** Sketch the complex numbers $0 + j2$ and $-5 - j2$.

**Solution:**

The alternative geometric representation for complex numbers is the polar form. Since a complex number can be represented as a point in the real-imaginary plane, and points in this plane can also be represented in polar coordinates, a complex number can be represented in polar form by $C = Z \angle \theta$ where $Z$ is the distance, or magnitude, from the origin, and $\theta$ is the angle measured counterclockwise from the positive real axis. Wow, that was a long sentence!

So, for example, $C = 10 \angle 53.13^\circ$ would be plotted as:

$C = 10 \angle 53.13^\circ$

(polar form)
Example: Sketch the complex numbers $2 \angle 20$, $3 \angle -120$ and $-2 \angle 45$.

Solution:

Conversion Between Forms We often need to convert between rectangular and polar forms.

To convert between forms where

- $C = a + jb$ (rectangular form)
- $C = C \angle \theta$ (polar form)

apply the following relations

- $a = C \cos \theta$
- $b = C \sin \theta$
- $C = \sqrt{a^2 + b^2}$
- $\theta = \tan^{-1} \frac{b}{a}$

Example: Convert $(5 \angle 60)$ to rectangular form.

Solution:

Example: Convert $6 + j7$ to polar form.

Solution:

Example: Convert $-4 + j4$ to polar form.

Solution:

Example: Convert $(5 \angle 220)$ to rectangular form.

Solution:

Properties of $j$ $j = \sqrt{-1}$ has a number of fascinating properties:

- $j^2 = (\sqrt{-1})(\sqrt{-1}) = -1$
- $\frac{1}{j} \cdot \frac{j}{j} = \frac{j}{j^2} = -j$
Addition and Subtraction of Complex Numbers  This is easiest to perform in rectangular form. We simply add/subtract the real and imaginary parts separately. For example,

\[(6 + j12) + (7 + j2) = (6 + 7) + j(12 + 2) = 13 + j14\]

\[(6 + j12) - (7 + j2) = (6 - 7) + j(12 - 2) = -1 + j10\]

Multiplication and Division of Complex Numbers  This is easiest to perform in polar form.

For multiplication: multiply magnitudes and add the angles

\[(6 \angle 70) \cdot (2 \angle 30) = 6 \cdot 2 \angle (70 + 30) = 12 \angle 100\]

For division: Divide the magnitudes and subtract the angles

\[\frac{6 \angle 70}{2 \angle 30} = \frac{6}{2} \angle (70 - 30) = 3 \angle 40\]

Two last points about complex numbers:

The reciprocal of \( C = \angle \theta \), is \( \frac{1}{C} = \frac{1}{\angle \theta} \)

The conjugate of \( C \) is denoted \( C^* \), and has the same real value but the opposite imaginary part:

\[ C = a + jb = C \angle \theta \]
\[ C^* = a - jb = C \angle -\theta \]

Important  You need to become very comfortable with doing complex arithmetic on your calculator!!!

Let’s take a moment for you to do the following problems on your calculator. For each problem, the answers are shown. Raise your hand if you are baffled (I mean baffled by these problems—not just in general).

\[(3 - i4) + (10 \angle 44) = \]
Convert this to rectangular:
ans(1)►rect

\[(22000+i13)/(3 \angle -17) = \]
Convert this to rectangular:
ans(1)►rect

Convert 95-12j to polar:
(95-12i)►polar
Example:

Convert these rectangular forms to polar:

a. \( Z_1 = 4 + j2 = \) ________________  
b. \( Z_2 = -6 + j3 = \) ________________

Convert these polar forms to rectangular:

c. \( Z_3 = 4 \angle 30 = \) ________________ 
   b. \( Z_4 = 6 \angle 70 = \) ________________

Example: Given: \( A = 1 + j1 \) and \( B = 2 - j3 \), determine \( A + B \), \( A - B \), \( A / B \) and \( A * B \).

Solution:

So, remember all that talk on page one about transforms….

To solve problems that involve sinusoids (such as AC voltages and currents) we use the phasor transform. That is, we transform sinusoids into complex numbers in polar form, solve the problem using complex arithmetic (as described above), and then transform the result back to a sinusoid.

So, first things first, how do we transform a sinusoid into a complex number in polar form? Here is how:

- Looking at the sinusoid, determine \( V_{pk} \) and the phase offset \( \theta \).
- Using \( V_{pk} \), determine \( V_{RMS} \) using the formula \( V_{RMS} = \frac{V_{pk}}{\sqrt{2}} \).
- The phasor is then \( V_{RMS} \angle \theta \).

Example: Express \( 100 \sin \omega t \) as a phasor.

Solution:

Example: Express \( 50 \sin (\omega t + 45^\circ) \) as a phasor.

Solution:
Phase Difference with Phasors  
Note that the waveform generated by the *leading phasor* leads the waveform generated by the *lagging phasor.*

![Phasor Diagram](image)

(a) $I_m$ leads $V_m$

(b) Therefore, $i(t)$ leads $v(t)$

Formulas from Trigonometry  
The following fun formulas from trigonometry are sometimes used to express phasors, particularly if we are dealing with signals expressed in cosines instead of sines.

\[
\begin{align*}
\cos(\omega t + \theta) &= \sin(\omega t + \theta + 90^\circ) \\
\sin(\omega t + \theta) &= \cos(\omega t + \theta - 90^\circ) \\
\cos(\omega t \pm 180^\circ) &= -\cos(\omega t) \\
\sin(\omega t \pm 180^\circ) &= -\sin(\omega t)
\end{align*}
\]

\[\therefore \cos(\omega t + 70^\circ) = \sin(\omega t + 160^\circ) = -\sin(\omega t - 20^\circ)\]

Example: Express $50\cos(\omega t + 45^\circ)$ as a phasor.

Solution:

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So, again, phasor representations can be viewed as complex number in polar form.

\[
E = E_m \angle \theta
\]

\[
e(t) = \sqrt{2}E_m \sin(\omega t + \theta)
\]

By replacing $e(t)$ with it’s phasor equivalent $E$, we have transformed the source from the time domain to the phasor domain.

**Important Notes**  
Peak values are only useful for time domain representations of signals. RMS values are the standard when dealing with phasor domain representations. If you need to represent something in the time domain, you will need to convert RMS to Peak voltage to obtain $E_m$

\[
E = V_{RMS} \angle \theta \quad \rightarrow \quad E_m = V_{RMS} \sqrt{2} \quad \rightarrow \quad e(t) = E_m \sin(2\pi ft + \theta)
\]
Example: Determine the phasor form of this waveform:

Solution:

Example: Given \( i(t) = 40 \sin(\omega t + 80^\circ) \) and \( v = -30 \sin(\omega t - 70^\circ) \), draw the phasor diagram, determine phase relationships, and sketch the waveforms.

Example: Given:

a. 100 Hz current source that leads 45° where \( I_{\text{RMS}} = 3.5 \text{ A} \).

b. 13 kHz current that lags 36° where \( I_{\text{PP}} = 36 \text{ mA} \).

REQD: Express these AC current sources as sinusoidal equations and as phasors. Graph these phasors.

Solution:
Example: Graph these complex numbers as phasors:
   a. $Z_1 = 4 + j2$
   b. $Z_2 = -6 + j3$
   c. $Z_3 = 10 - j6$

Solution:

Example: Given:
   $I_1 = 20 \sin (\omega t)$ mA.
   $I_2 = 10 \sin (\omega t + 90^\circ)$ mA.
   $i_3 = 30 \sin (\omega t - 90^\circ)$ mA.

Determine $i_R(t)$.

Solution:

Example: Find $e_m$ in the circuit below if $v_a = 50 \sin (377t + 30^\circ)$ and $v_b = 30 \sin (377t + 60^\circ)$. 
**Impedance**

**The Impedance Concept**  Impedance ($Z$) is the opposition that a circuit element presents to current in the phasor domain. If we view voltage and current as phasors (which are complex numbers in polar form), then the impedance is defined as

$$Z = \frac{V}{I} = \frac{V}{I} \angle \theta = Z \angle \theta$$

This lends itself to an interpretation of Ohm’s law for ac circuits:

$$V = IZ$$

In AC circuits, the impedance plays the same role that resistance ($R$) played in DC circuits.

Since impedance is a complex quantity it can also be expressed in rectangular form. In rectangular form, the impedance is made up of a real part, called the resistance $R$ (the same resistance that we know and love from DC!), and an imaginary part called the reactance $X$:

$$Z = R + jX$$  $(\Omega)$

Note that the unit of impedance is ohms.

You are undoubtedly right this very minute completely paralyzed by the searing burning question: “I know where resistance comes from: resistors! But where does ‘reactance’ come from? Where? What the heck is reactance? I simply cannot go on in life until this question is answered.”

Since we do not want to see our midshipmen paralyzed, we will spend the next hour answering this burning question so that you can move forward with your lives. Lucky you.

**Resistance and Sinusoidal AC**  For a purely resistive circuit, current and voltage are in phase.

$$Z_R = \frac{V_R}{I} = \frac{V_R \angle \theta}{I \angle \theta} = \frac{V_R}{I} \angle 0^\circ = R \angle 0^\circ = R$$

(b) $i_R = v_p/R$. Therefore $i_R$ is a sine wave also
Example: Using phasors, find the voltage $v$ in the circuit below (the “$q$” in the equation should be $\omega$—even books in their 12th edition can still have the occasional typo!).

Solution:

\[ i = 4 \sin(\omega t + 30^\circ) \]

Note the phasor diagram for the preceding example. In a purely resistive circuit, the current and the voltage are in phase.
**Inductance and Sinusoidal AC** Suppose that the current through an inductor is sinusoidal:

\[ i_L(t) = I_m \sin(\omega t) \]

and thus, the current expressed as a phasor is:

\[ I_L = \frac{I_m}{\sqrt{2}} \angle 0^\circ \]

What is the voltage across this inductor? Recall the voltage-current relationship for an inductor:

\[ v_L = L \frac{di_L}{dt} = L \frac{d}{dt} (I_m \sin \omega t) = \omega LI_m \cos \omega t = \omega LI_m \sin(\omega t + 90^\circ) \]

So, as a phasor, the voltage across the inductor is:

\[ V_L = \frac{\omega LI_m}{\sqrt{2}} \angle 90^\circ \]

Thus, the impedance, which is the phasor voltage divided by the phasor current is

\[ Z_L = \frac{V_L}{I_L} = \frac{\omega LI_m}{\frac{I_m}{\sqrt{2}} \angle 0^\circ} = \omega L \angle 90^\circ \quad (\Omega) \]

The impedance of an inductor can be written as a complex number (in polar or rectangular form):

\[ Z_L = \omega L \angle 90^\circ = j \omega L \quad (\Omega) \]

**MEMORIZE IT.**

Since an ideal inductor has no real resistive component, this means the reactance of an inductor is the pure imaginary part:

\[ X_L = \omega L \quad (\Omega) \]

It should be noted that for a purely inductive circuit, the voltage leads the current by 90°.

**Variation with Frequency:** Since \( X_L = \omega L = 2\pi f L \), inductive reactance is directly proportional to frequency. For the extreme case of 0 Hz (DC), the inductor looks like a short circuit!
Example: Using the phasor transform, find the voltage $v$ in the circuit below.

Solution:

Capacitance and Sinusoidal AC Suppose that the voltage across a capacitor is sinusoidal:

$$v_C(t) = V_m \sin(\omega t)$$

and thus, the voltage expressed as a phasor is:

$$V_C = \frac{V_m}{\sqrt{2}} \angle 0^\circ$$

What is the current through this capacitor? Recall the voltage-current relationship for a capacitor:

$$i_c = C \frac{dv_C}{dt} = C \frac{d}{dt}(V_m \sin \omega t) = \omega CV_m \cos \omega t = \omega CV_m \sin(\omega t + 90^\circ)$$

So, as a phasor, the current through the capacitor is:

$$I_c = \frac{\omega CV_m}{\sqrt{2}} \angle 90^\circ$$

Thus, the impedance, which is the phasor voltage divided by the phasor current is

$$Z_c = \frac{V_c}{I_c} = \frac{\frac{V_m}{\sqrt{2}} \angle 0^\circ}{\frac{\omega CV_m}{\sqrt{2}} \angle 90^\circ} = \frac{1}{\omega C} \angle -90^\circ \text{ (}\Omega\text{)}$$
Impedance can be written as a complex number (in polar or rectangular form):

\[
Z_c = \left(\frac{1}{\omega C}\right) \angle -90^\circ = -j \left(\frac{1}{\omega C}\right) \quad (\Omega)
\]

**MEMORIZE IT.**

Since a capacitor has no real resistive component, this means the reactance of a capacitor is the pure imaginary part:

\[
X_c = \left(\frac{1}{\omega C}\right)
\]

It should be noted that, for a purely capacitive circuit current leads the voltage by 90º.

Variation with Frequency  Since \( X_c = \frac{1}{\omega C} = \frac{1}{2\pi fC} \), capacitive reactance is inversely proportional to frequency. As an extreme case, if \( f = 0 \) Hz (DC), the capacitor looks like an open circuit!

Example:  Find the voltage \( v \) in the circuit below.

**Solution:**

Impedance and AC Circuits solution technique
1. Transform the time domain currents and voltages into phasors
2. Calculate the impedances for circuit elements
3. Perform all calculations using complex math in the phasor domain
4. Transform the resulting phasors back to time domain if required

Example: Consider the circuit shown which as two resistors $R_1=10 \, \text{k}\Omega$ and $R_2=12.5 \, \text{k}\Omega$ in series. The current is $i(t) = 14.7 \sin(\omega t + 39^\circ) \, \text{mA}$.

(a) Compute $V_{R1}$ and $V_{R2}$
(b) Compute $V_T = V_{R1} + V_{R2}$
(c) Calculate $Z_T$
(d) Compare $V_T$ to the results of $V_T = IZ_T$

Solution:
Example: For the inductive circuit shown, $v_L = 40 \sin (\omega t + 30^\circ)$ V, $f = 26.53$ kHz and $L = 2$ mH. Determine $V_L$ and $I_L$. Graph $i_L$ and $v_L$.

Solution:

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Example: For the inductive circuit, $v_L = 40 \sin (\omega t + \Theta)$ V and $i_L = 250 \sin (\omega t + 40^\circ)$ μA. The frequency is $f = 500$ kHz.

Determine:
(a) $L$
(b) $\Theta$

Solution:
Example: For the capacitive circuit, \( v_C = 3.6 \sin (\omega t - 50) \) V, \( C = 1.29 \mu F \) and \( f = 12 \) kHz. Determine \( V_C \) and \( I_C \) and then plot \( V_C \) and \( I_C \) as phasors.

Solution:

Example: For the capacitive circuit:
\[
v_C = 362 \sin (\omega t - 33^\circ) \text{ V}, \quad i_C = 94 \sin (\omega t + 57^\circ) \text{ mA} \quad \text{and} \quad C = 2.2 \mu F.
\]
Determine the frequency.

Solution: