Basic Functions Sketch the functions listed below; use their graphs to fill in the blanks:

$\lim_{x\to\infty} \frac{1}{x} =$

$\lim_{x\to\infty} \ln x =$

$\lim_{x\to\infty} \arctan x =$

In general

$\lim_{x\to\infty} e^{x} =$ For $a > 1$, $\lim_{x\to\infty} a^{x} =$

$\lim_{x\to\infty} e^{0.1\,x} =$ For $k >$, $\lim_{x\to\infty} e^{kx} =$

$\lim_{x\to\infty} 0.7^{x} =$ For $0 < a < 1$,

$\lim_{x\to\infty} e^{-2\,x} =$ For $k < 0$, $\lim_{x\to\infty} e^{kx} =$

Ratios of Basic Functions

If $P(x)$ is a polynomial or if $P(x)$ is a positive power of x (such as $P(x) = x^{24} - 3x$ or $P(x) = \sqrt[4]{x}$) then

$\lim_{x\to\infty} \frac{\ln x}{P(x)} = 0$ and for $a > 1$, $\lim_{x\to\infty} \frac{P(x)}{a^{x}} = 0$.

Colloquially: at infinity logarithms grow slower than powers, and powers grow slower than exponentials.

Combinations of Basic Functions: how to quickly compute $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ for functions f, g satisfying $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} g(x) = \infty$.

First, identify the fastest growing term within $f(x)$ and factor it (it will be a forced factor, not a common one); identify the fastest growing term within g and factor it.

Second: Simplify the ratio of the forced factors from step 1 and evaluate its limit using basic function techniques. Evaluate separately the limit of the remaining factor from f, and form g; they should be finite, non-zero numbers. Combine your three limits to finish.

Example: find $\lim_{x\to\infty} \frac{8 \cdot 4^{x} + 5^{x}}{3^{x} + \sqrt{x}}$.

Step 1: The fastest growing among the exponentials $8 \cdot 4^x$ and 5^x is 5^x. Write $8 \cdot 4^x + 5^x = 5^x(\dfrac{8 \cdot 4^x}{5^x} + 1)$.

The fastest growing among 3^x and $\sqrt{x}$ is the exponential 3^x. Write $3^x + \sqrt{x} = 3^x(1 + \dfrac{\sqrt{x}}{3^x})$.

Step 2: write

$$\lim_{x\to\infty} \frac{8 \cdot 4^x + 5^x}{3^x + \sqrt{x}} = \lim_{x\to\infty} \frac{5^x}{3^x} \frac{\frac{8 \cdot 4^x}{5^x} + 1}{\frac{\sqrt{x}}{3^x} + 1}.$$

Compute separately the simpler limits:

$$\lim_{x\to\infty} \frac{5^x}{3^x} = \lim_{x\to\infty} \left(\frac{5}{3}\right)^x =$$

$$\lim_{x\to\infty} \frac{8 \cdot 4^x}{5^x} + 1 = 1 + 8 \lim_{x\to\infty} \left(\frac{4}{5}\right)^x =$$

and

$$\lim_{x\to\infty} \frac{\sqrt{x}}{3^x} + 1 =$$

Conclude $\displaystyle\lim_{x\to\infty} \frac{8 \cdot 4^x + 5^x}{3^x + \sqrt{x}} =$ Answer: ∞.

Example: find $\displaystyle\lim_{x\to\infty} \frac{3x^4 + 5\ln x - 12}{(x+4)(6 + x^3 + 5x)}$.

Step 1: factor x^4 or $3x^4$ (your choice) out of the numerator (remember that powers grow faster than logs)

$$3x^4 + 5\ln x - 12 = x^4(3 + 5\frac{\ln x}{x^4} - \frac{12}{x^4})$$

Factor x out of $(x+4)$ and factor x^3 out of $(6 + x^3 + 5x)$ to get

$$(x+4)(6 + x^3 + 5x) = x(1 + \frac{4}{x})x^3(1 + \frac{5}{x^2} + \frac{6}{x^3}) = x^4\left[(1 + \frac{4}{x})(1 + \frac{5}{x^2} + \frac{6}{x^3})\right].$$

We write

$$\frac{3x^4 + 5\ln x - 12}{(x+4)(x^3+5)} = \frac{x^4}{x^4} \frac{3 + 5\frac{\ln x}{x^4} - \frac{12}{x^4}}{(1 + \frac{4}{x})(1 + \frac{5}{x^2} + \frac{6}{x^3})}.$$

Compute separately the limits at infinity of

$$\frac{x^4}{x^4}$$

$$3 + 5\frac{\ln x}{x^4} - \frac{12}{x^4}$$

and

$$(1 + \frac{4}{x})(1 + \frac{5}{x^2} + \frac{6}{x^3}).$$

Combine them and finish the problem.

(the limits are 1, 3, and 1. Answer: 1(3)/1=3)