Proof 1: Prove that the derivative of the function $f(x) = e^x$ is $\frac{df}{dx} = e^x$. You may use the following limit results:

$$\lim_{h\to 0}\frac{e^h-1}{h} = 1, \quad \lim_{h\to 0}\frac{\cos h - 1}{h} = 0, \text{and} \quad \lim_{h\to 0}\frac{\sin h}{h} = 1$$

Proof: According to the definition of the derivative,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x e^h - e^x}{h} =$$

$$\lim_{h\to 0}\frac{e^x(e^h-1)}{h} = e^x \lim_{h\to 0}\frac{e^h-1}{h} = e^x(1) = e^x.$$

Proof 2: Prove that the derivative of the function $g(x) = \sin x$ is $\frac{dg}{dx} = \cos x$. You may use the following limit results:

$$\lim_{h\to 0}\frac{e^h-1}{h} = 1, \quad \lim_{h\to 0}\frac{\cos h - 1}{h} = 0, \text{ and } \quad \lim_{h\to 0}\frac{\sin h}{h} = 1$$

Proof: According to the definition of the derivative,

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h + \sin h\cos x - \sin x}{h}.$$

Here we used the trig identity of $\sin(x+y) = \sin x\cos y + \sin y\cos x$. We get:

$$g'(x) = \lim_{h\to 0}\left[\frac{\sin x\cos h - \sin x}{h} + \frac{\sin h\cos x}{h}\right] = \lim_{h\to 0}\frac{\sin x(\cos h - 1)}{h} + \cos x \lim_{h\to 0}\frac{\sin h}{h}.$$

Applying the $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$ and $\lim_{h\to 0} \frac{\sin h}{h} = 1$, we get

$$g'(x) = (\sin x)(0) + (\cos x)(1) = \cos x.$$

Proof 3: State and prove Product Rule for derivatives.

State: If f and g are functions differentiable at a, then the product fg is differentiable at a and

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

Proof: According to the definition of the derivative,

$$(fg)'(a) = \lim_{x\to a}\frac{f(x)g(x)-f(a)g(a)}{x-a} = \lim_{x\to a}\frac{f(x)g(x)-f(x)g(a)+f(x)g(a)-f(a)g(a)}{x-a} =$$

$$= \lim_{x\to a}\left[\frac{f(x)g(x)-f(x)g(a)}{x-a} + \frac{f(x)g(a)-f(a)g(a)}{x-a}\right] = \lim_{x\to a}\left[\frac{f(x)[g(x)-g(a)]}{x-a} + \frac{g(a)[f(x)-f(a)]}{x-a}\right] =$$

$$= \lim_{x\to a} f(x) \lim_{x\to a}\frac{[g(x)-g(a)]}{x-a} + g(a)\lim_{x\to a}\frac{[f(x)-f(a)]}{x-a}.$$

Since f is differentiable at a, it is continuous at a, thus $\lim_{x\to a} f(x) = f(a)$.

According to the derivative definition, the remaining two limits (the fractions) represent $g'(a)$ and $f'(a)$, respectively. Thus we showed that fg is differentiable at a, and $(fg)'(a) = f(a)g'(a) + g(a)f'(a)$. Reorder the terms to conclude $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.