SM 121: TAYLOR POLYNOMIALS HANDOUT

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Linear Approximation

The "best linear approximation to the function $f(x)$ near $x = a$" is

$$L(x) = f(a) + f'(a)(x - a).$$

It is "the best" because $L(a) = f(a)$ and $L'(a) = f'(a)$. That is, $y = f(x)$ and $y = L(x)$ pass through the same point with the same slope, when $x = a$. We also call $L(x)$ the "degree 1 Taylor polynomial for $f(x)$ centered at $x = a$", denoted $T_1(x)$. Thus

$$\boxed{T_1(x) = f(a) + \frac{f'(a)}{1!}(x - a)}$$

Example 1. *Find the degree 1 Taylor polynomial for $f(x) = \sin(x)$ centered at $x = 0$, and use it to estimate $\sin(0.2)$. How does your estimate compare with the true (calculator) value?*

Solution. Degree 1 Taylor polynomial:

$$T_1(x) = f(0) + \frac{f'(0)}{1!}(x - 0)$$

$$f(x) = \sin(x) \implies f(0) = \sin(0) = 0$$

$$f'(x) = \cos(x) \implies f'(0) = \cos(0) = 1$$

$$T_1(x)0 + \frac{1}{1!}(x - 0) = x$$

Estimate:

$$\sin(0.2) = f(0.2) \approx T_1(0.2) = 0.2$$

True value:

$$\sin(0.2) \approx 0.19866933$$

□

Date: Fall 2017.

Quadratic Approximation

The "best quadratic approximation to the function $f(x)$ near $x = a$" is

$$Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2.$$

It is "the best" because $Q(a) = f(a)$, $Q'(a) = f'(a)$, and $Q''(a) = f''(a)$. That is, $y = f(x)$ and $y = Q(x)$ pass through the same point with the same slope and the same concavity, when $x = a$. We also call $Q(x)$ the "degree 2 Taylor polynomial for $f(x)$ centered at $x = a$", denoted $T_2(x)$. Thus

$$\boxed{T_2(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2}$$

Example 2. *Find the degree 2 Taylor polynomial for $f(x) = \cos(x)$ centered at $x = 0$, and use it to estimate $\cos(0.3)$. How does your estimate compare with the true (calculator) value?*

Solution. Degree 2 Taylor polynomial:

$$T_2(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2$$

$$f(x) = \cos(x) \implies f(0) = \cos(0) = 1$$

$$f'(x) = -\sin(x) \implies f'(0) = -\sin(0) = 0$$

$$f''(x) = -\cos(x) \implies f''(0) = -\cos(0) = -1$$

$$T_2(x) = 1 + \frac{0}{1!}(x-0) + \frac{-1}{2!}(x-0)^2 = 1 - x^2/2$$

Estimate:

$$\cos(0.3) = f(0.3) \approx T_2(0.3) = 0.955$$

True value:

$$\cos(0.3) \approx 0.955336489$$

□

Cubic Approximation

The "best cubic approximation to the function $f(x)$ near $x = a$" is

$$C(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3.$$

It is "the best" because $C(a) = f(a)$, $C'(a) = f'(a)$, $C''(a) = f''(a)$, and $C'''(a) = f'''(a)$. We also call $C(x)$ the "degree 3 Taylor polynomial for $f(x)$ centered at $x = a$", denoted $T_3(x)$. Thus

$$\boxed{T_3(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3}$$

Example 3. *Find the degree 3 Taylor polynomial for $f(x) = \sqrt{x}$ centered at $x = 4$, and use it to approximate $\sqrt{3.7}$.*

Solution. Degree 3 Taylor polynomial:

$$T_3(x) = f(4) + \frac{f'(4)}{1!}(x-4) + \frac{f''(4)}{2!}(x-4)^2 + \frac{f'''(4)}{3!}(x-4)^3$$

$$f(x) = x^{1/2} \implies f(4) = 4^{1/2} = 2$$

$$f'(x) = \frac{1}{2}x^{-1/2} \implies f'(4) = \frac{1}{2}\cdot 4^{-1/2} = \frac{1}{4}$$

$$f''(x) = -\frac{1}{4}x^{-3/2} \implies f''(4) = -\frac{1}{4}\cdot 4^{-3/2} = -\frac{1}{32}$$

$$f'''(x) = \frac{3}{8}x^{-5/2} \implies f'''(4) = \frac{3}{8}\cdot 4^{-5/2} = \frac{3}{256}$$

$$\begin{aligned} T_3(x) &= 2 + \frac{1/4}{1!}(x-4) + \frac{-1/32}{2!}(x-4)^2 + \frac{3/256}{3!}(x-4)^3 \\ &= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 \end{aligned}$$

Estimate:

$$\sqrt{3.7} = f(3.7) \approx T_3(3.7) \approx 1.923541016$$

True value:

$$\sqrt{3.7} = 1.923538406$$

□

Higher-Order Approximation

For any n, there is a degree n Taylor polynomial for $f(x)$ centered at $x = a$, denoted $T_n(x)$. It is the "best degree n approximation to the function $f(x)$ near $x = a$", because $T_n(a) = f(a)$, $T_n'(a) = f'(a)$, $T_n''(a) = f''(a)$, ..., $T_n^{(n)}(a) = f^{(n)}(a)$. Predictably,

$$\boxed{T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

Exercises

Question 1. *Find the degree 3 Taylor polynomial for $f(x) = e^x$ centered at $x = 0$, and use it to approximate $\sqrt{e}$. How does your estimate compare with the true (calculator) value?*

Question 2. *Find the degree 3 Taylor polynomial for $f(x) = \tan(x)$ centered at $x = 0$, and use it to approximate $\tan(1/2)$. How does your estimate compare with the true (calculator) value?*

Question 3. *Find the degree 3 Taylor polynomial for $f(x) = x^{1/3}$ centered at $x = 1$, and use it to approximate $\sqrt[3]{2}$. How does your estimate compare with the true (calculator) value?*

Question 4. *Find the degree 3 Taylor polynomial for $f(x) = \ln(x)$ centered at $x = 1$, and use it to approximate $\ln(1.2)$. How does your estimate compare with the true (calculator) value?*

Question 5. *Find the degree 3 Taylor polynomial for $f(x) = \tan^{-1}(x)$ centered at $x = 0$, and use it to estimate $\tan^{-1}(1/3)$. How does your estimate compare with the true (calculator) value?*

Question 6. *Find the degree 3 Taylor polynomial for $f(x) = x^x$ centered at $x = 1$, and use it to estimate $1.1^{1.1}$. How does your estimate compare with the true (calculator) value?*