← version 4

V1?

1.  B

---

2. $\cancel{b}$ $4+1+9 \neq 0$ D

(d)

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3. $\overrightarrow{MN} = -(2,3,0)+(0\,0\,5)$ D

$MN = \langle -2, -3, 5 \rangle$

---

4. $\vec{r}\,' = \langle \frac{1}{t}, 0, 3t^2 \rangle$, $t=1$

$\vec{r}\,' = \langle 1, 0, 3 \rangle$

$\cancel{a}$ $\cancel{b}$ $\cancel{c}$ $\cancel{c}$ D

---

5. $\vec{a} = \langle 4\cos\frac{1}{2}t, 3\sin t \rangle$ E

$v = \langle 8\sin\frac{1}{2}t, -3\cos t \rangle + c$

$v(0)=0$ $\vec{V} = \langle 8\sin\frac{1}{2}t, 3-3\cos t \rangle$

$v(\pi) = \langle 8, 6 \rangle$

$\sqrt{64+36} = 10$

---

6. circles E

---

7. $f_x = \frac{1}{z}\sin(yz)$ B

$f_{xy} = \frac{1}{z}\cos(yz)\cdot z = \cos(yz)$

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8. $f = 2xe^z + 3y$ C

$f_x = 2e^z = 2$

$f_y = 3$ $f_z = 2xe^z = 2(3) = 6$

$|\langle 2, 3, 6 \rangle| = \sqrt{4+9+36} = 7$

Final Exam
SM221 Dec 15, 2018

9. $f_x = x + 3y - 10$ (D)

$f_y = 3x - y$

$A(0,2)$ $f_x(A) \neq 0$

$B(1,3)$ $f_x = 0$

$f_y = 0$

$f_{xx} = 1$ $f_{xy} = 3$ $f_{yy} = -1$

$D = 1(-1) - 9 < 0$ saddle

---

10. $z_x = -2(0.05)x = -0.1$ (C)

$+x$ means east

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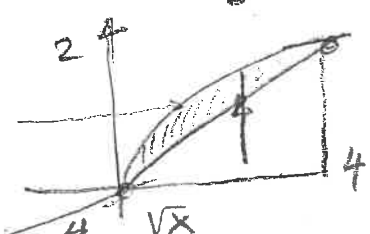
11. $y^2 \le x \le 2y$

$\frac{1}{2}x \le y \le \sqrt{x}$

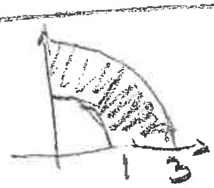


$y = \frac{1}{2}x$

$$\int_0^4 \int_{\frac{1}{2}x}^{\sqrt{x}} f \, dy \, dx$$ (B)

(A)

12.



$f = \frac{r\cos\theta}{r^2}$

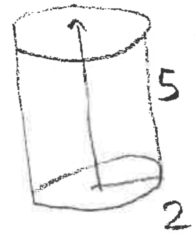
$$\int_0^{\pi/2} \int_1^3 \cos\theta \, dr \, d\theta = \int_0^{\pi/2} 2\cos\theta \, d\theta$$

$$= 2\sin\theta \Big|_0^{\pi/2} = 2$$

13 circle

$\cancel{a}$ $\cancel{b}$ c $\cancel{d}$ $\cancel{e}$

c



A

14

$f = r$

$$\int_0^{2\pi}\int_0^{2}\int_0^{5} r^2\,dz\,dr\,d\theta$$

E

15.

$\operatorname{div} F = -2y + 2y + 4 = 4$

$\iiint 4\,dV = 4(\pi 2^2)\cdot 5 = 80\pi$

---

16. $x = t \quad y = e^t \quad 0 \le t \le 1$

$r' = \langle 1, e^t \rangle$

A

$\vec{F} = \langle e^t e^t, e^t \rangle$

$$\int_0^1 (e^{2t} + e^{2t})\,dt = \int_0^1 2e^{2t}\,dt = e^{2t}\Big|_0^1$$

$= e^2 - 1$

alternate

$F = \nabla f, \quad f = ye^x$

$f(0,1) = 1$

$f(1,e) = e^2$

$\boxed{\text{Answer } e^2 - 1}$

---

17. $\int \nabla f \cdot dr = f(B) - f(A)$

C

$t = 1 \quad A(3,1) \quad f = 10$

$t = 4 \quad B(0,4) \quad f = 16$

$16 - 10 = 6$

18. $z = y^2 - x + 2$ (A)

$d\vec{S} = \langle 1, -2y, 1 \rangle \, dx\,dy$

$z_x = -1 \quad z_y = 2y$

$\iint \langle z, 0, x - y^2 \rangle \langle 1, -2y, 1 \rangle \, dA$

$= \iint z - y^2 + x \, dA = \iint (y^2 - x + 2 - y^2 + x) \, dA$

$= \iint 2 \, dA = 2(1)(1) = 2$

---

19. $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 4 - 3 = 1$ (D)

$\iint 1 \, dA = \frac{1}{2}(3)(2) = 3$

---

20 $z = x^2 - 2x + y^2 - y \quad P(1,3,5)$

$z_x = 2x - 2 \Big|_P = 0$ (A)

$z_y = 2y - 1 = 5$

$z = 5 + 0(x-1) + 5(y-3)$

$-5(y-3) + (z-5) = 0$

(21) $\vec{OA} = \vec{DB}$

$\vec{OD} = \vec{AB}$

a. $\vec{OB} = \vec{OA} + \vec{OD} = \langle 10, 2 \rangle$

b. $T_x = 8 - \frac{2x}{49} y^3$

$T_y = -30 - \frac{3y^2}{49} x^2$

$\nabla T(0,0) = \langle 8, -30 \rangle$

$\vec{OD} = \langle 7, 4 \rangle$

$|\vec{OD}| = \sqrt{49+16} = \sqrt{65}$

$D_{\vec{OD}} T(0\,0) = \frac{\langle 8, -30 \rangle \cdot \langle 7, 4 \rangle}{\sqrt{65}}$

$D_{\vec{OD}} T = \frac{56-120}{\sqrt{65}} = \frac{-64}{\sqrt{65}} \approx -7.94$

c. $\vec{OB} = \vec{OA} + \vec{OD} = \langle 10, 2 \rangle$ (moved to a)

c) ~~d.~~ $\cos\theta = \frac{\langle 3, -2 \rangle \langle 10, 2 \rangle}{\sqrt{9+4}\ \sqrt{104}}$

$\cos\theta = \frac{26}{13 \cdot 2\sqrt{2}} = \frac{1}{\sqrt{2}} \approx 0.707$

$\theta = \pi/4$ $\qquad \theta \approx 0.7855$ rad

(22) See comment at bottom

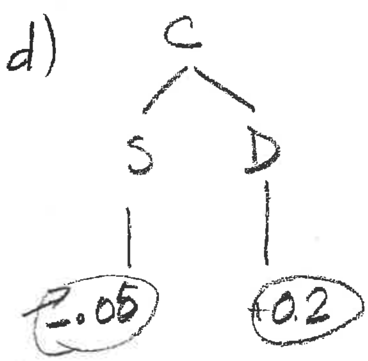
a) $\dfrac{\partial C}{\partial S} = \dfrac{1496 - 1489.8}{35-30} = 1.24 \dfrac{m/s}{g/\ell}$

b) If salinity is 35 g/ℓ, at water depth 100 m, an increase in salinity produces an increase of 1.24 m/s in the speed of sound for each 1 g/ℓ increase in salinity

c) $\dfrac{\partial C}{\partial D} = \dfrac{1496.8 - 1496}{1.5-1}$ or $\dfrac{1496.8 - 1495.2}{1.5-0.5} = 1.6 \dfrac{m/s}{km}$

(or 1.6 meters per hundred seconds or 1.6 cm/s)

d)



$\dfrac{dS}{dt} = -0.05 \dfrac{g/\ell}{h}$

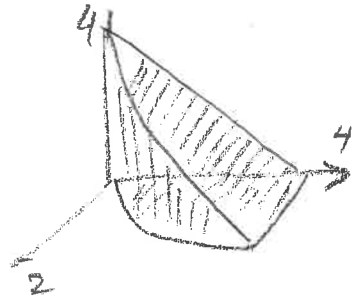
$\dfrac{dD}{dt} = 0.2 \; km/h$

$\dfrac{dC}{dt} = \dfrac{\partial C}{\partial S}\dfrac{dS}{dt} + \dfrac{\partial C}{\partial D}\dfrac{dD}{dt} = 1.24(-0.05) + 1.6(.2)$

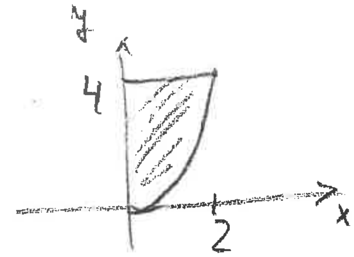
$\dfrac{dC}{dt} = -0.062 + 0.32 = 0.258 \dfrac{m/s}{hour}$

Note: the chart gives $C = C_0 + 1.24S + 1.6D$, so even if they use central differences/one-sided differences, $\frac{\partial C}{\partial D}$, $\frac{\partial C}{\partial S}$ won't change.

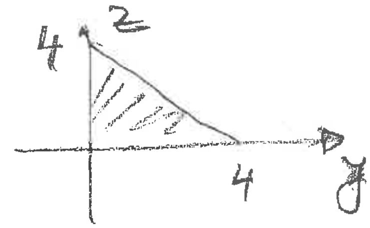
(23)



$y = x^2, \; x = \sqrt{y}$
$z = 4 - y$



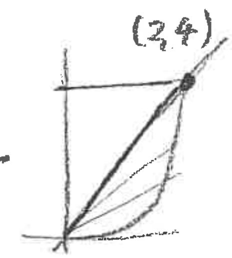
$\rho = \frac{1}{4-y}$



a) (i) $\displaystyle\int_0^4 \int_0^{\sqrt{y}} \int_0^{4-y} \frac{1}{4-y}\, dz\, dx\, dy$

(ii) $\displaystyle\int_0^2 \int_{x^2}^4 \int_0^{4-y} \frac{1}{4-y}\, dz\, dy\, dx$

iii) $\displaystyle\int_0^4 \int_0^{4-y} \int_0^{\sqrt{y}} \frac{1}{4-y}\, dx\, dz\, dy$



<u>bad idea</u>: $\displaystyle\int_0^{\arctan 2} \int_0^{\frac{\sin\theta}{\cos^2\theta}} \int_0^{4-r\sin\theta} \frac{1}{4-r\sin\theta}\, r\, dz\, dr\, d\theta +$

($y = x^2$ becomes $r\sin\theta = r^2\cos^2\theta$, $r = \frac{\sin\theta}{\cos^2\theta}$)

$\displaystyle + \int_{\arctan 2}^{\pi/2} \int_0^{\frac{4}{\sin\theta}} \int_0^{4-r\sin\theta} \frac{1}{4-r\sin\theta}\, r\, dz\, dr\, d\theta$

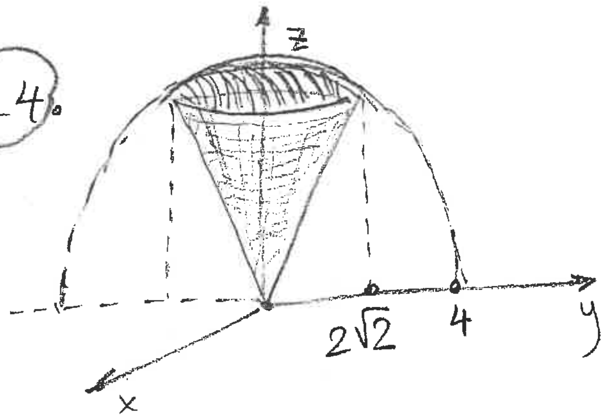
b) (i) = (ii) to start $\displaystyle\int_0^4 \int_0^{\sqrt{y}} 1\, dx\, dy = \int_0^4 \sqrt{y}\, dy = \frac{2}{3} y^{3/2} \Big|_0^4 = \frac{16}{3}$

$\displaystyle\int_0^2 \int_{x^2}^4 1\, dy\, dx = \int_0^2 (4 - x^2)\, dx = 8 - \frac{1}{3}\cdot 8 = \frac{16}{3}$

(iii) $\displaystyle\int_0^4 \int_0^{4-y} \sqrt{y}\, \frac{1}{4-y}\, dz\, dy = \int_0^4 \sqrt{y}\, dy = \frac{2}{3}\frac{3}{2} y^{3/2} \Big|_0^4 = \frac{16}{3}$

24.

a)



b) top sphere $x^2+y^2+z^2=16$, radius 4

bottom cone $z^2=x^2+y^2$ or $z=\sqrt{x^2+y^2}$

(or half cone)

c) $$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{4} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

d) $r^2+r^2=16$

$r^2=8$

$r=2\sqrt{2}$ or $\sqrt{8}$

$r^2+z^2=16$

$z=\sqrt{16-r^2}$

$$\int_0^{2\pi}\int_0^{2\sqrt{2}}\int_r^{\sqrt{16-r^2}} r \, dz \, dr \, d\theta$$

note: $dr\,dz\,d\theta$ is much harder.

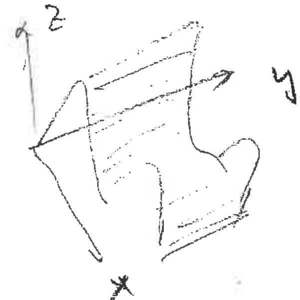
25. $\vec{F} = \langle -y, z-y, 2021^{2022z} \rangle$

a) $\text{curl } F = \begin{vmatrix} i & j & k \\ dx & dy & dz \\ -y & z-y & 2021^{2022z} \end{vmatrix} = i(0-1) - j(0-0) + k(0+1)$

$\text{curl } \vec{F} = \langle -1, 0, 1 \rangle$

b) $z = 1 + \cos x$, cylinder (no y)

first sketch



c) Stokes $\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times F) \cdot d\vec{S}$ third integral

d) $\iint_S \langle -1, 0, 1 \rangle \cdot d\vec{S}$

graph $z = 1 + \cos x$

$z_x = -\sin x$

$z_y = 0$

$d\vec{S} = \pm \langle \sin x, 0, 1 \rangle \, dx\,dy$

choose + / points up

$$\int_0^2 \int_{-\pi}^{3\pi} \langle -1, 0, 1 \rangle \cdot \langle \sin x, 0, 1 \rangle \, dx\,dy = \int_0^2 \int_{-\pi}^{3\pi} (1 - \sin x)\, dx\,dy$$

$$= \int_0^2 \left(x - \cos x \Big|_{x=-\pi}^{x=3\pi} \right) dy = \int_0^2 (4\pi - 0)\, dy = 8\pi$$

(26)

$$\frac{\partial f}{\partial x} = 1+y^2 \quad \rightsquigarrow f(x,y,z) = x + y^2 x + g(y,z)$$

$$\frac{\partial f}{\partial y} = 2xy + e^{3z} \qquad 2yx + \frac{\partial g}{\partial y} = 2xy + e^{3z}$$

$$\frac{\partial f}{\partial z} = 3ye^{3z} \qquad \frac{\partial g}{\partial y} = e^{3z}$$

$$g(y,z) = ye^{3z} + h(z)$$

$$\frac{\partial}{\partial z}\left(x+y^2x+ye^{3z}+h\right) = 3ye^{3z}$$

$$h'(z) = 0, \quad h = \text{constant}$$

$$\boxed{f(x,y,z) = x + y^2x + ye^{3z} + c}$$

Verify
$f_x = 1+y^2$
$f_y = 2yx + e^{3z}$
$f_z = 3ye^{3z}$

Note: students may use

$$f = \int(1+y^2)\,dx = x + xy^2 + g_1(y,z)$$

$$f = \int(2xy+e^{3z})\,dy = xy^2 + ye^{3z} + g_2(x,z)$$

$$f = \int 3ye^{3z}\,dz = ye^{3z} + g_3(x,y), \quad \text{and guess}$$

$$f(x,y,z) = x + xy^2 + ye^{3z} + C.$$

If they do, they need to check this f works