III. Example 1: R-L DC Circuit

Physical characteristics of the circuit: 60 volt DC battery connected in series with a 4 henry inductor and a 12 ohm resistor; current flows when the open switch is closed. (Note: This is Example #2 on p. 515 and #3 on p. 524 of Stewart: *Calculus—Concepts and Contexts*, 2nd ed.)

**Task:** Write down the Initial Value Problem associated with this circuit and solve it for the current in order to answer the following questions.

[a] Describe in words how the current changes over time.
[b] What is the current 0.1 second after the switch is closed?
[c] At what time does the current equal half the steady-state current?
[d] What is the average current over the first five time units for this circuit?

**Solution:** By Kirchhoff’s laws we have \( E_L + E_R = EMF \) which, with \( E_L = L \cdot I'(t) \) and \( E_R = R \cdot I(t) \), translates into the following Initial Value Problem (for \( t \geq 0 \)):

\[
4 I'(t) + 12 I(t) = 60, \quad I(t) = 0 \text{ at } t = 0
\]

We can solve for \( I \) using the method of separation of variables.

**Outline of solution by separation of variables**

First, we will divide the ODE through by 4, replace \( I(t) \) by \( I \), and use the differential notation for derivatives:

\[
\frac{dI}{dt} + 3 I = 15
\]

Next, use algebra to rewrite this as

\[
\frac{dI}{15 - 3 I} = dt
\]

and integrate both sides to obtain

\[
-\frac{1}{3} \ln |15 - 3 I| = t + C
\]
which with the initial condition $I(0) = 0$ yields the circuit current

$$I(t) = 5 - 5 e^{-3t}, \quad t \geq 0$$

More details for all these steps may be found below, after the Answers.

**Answers:**

[a] Describe in words how the current changes over time.

The following graph shows how $I(t)$ increases from 0 at $t = 0$ toward an asymptotic limit 5 as $t$ increases:

$$\lim_{t \to \infty} I(t) = 5 - 5 \lim_{t \to \infty} e^{-3t} = 5 - 5(0) = 5$$

This asymptotic limit is called the *steady-state* current.

![R-L Circuit: current I(t)](image)

[b] What is the current 0.1 second after the switch is closed?

$$I(0.1) = 5 - 5 e^{-3(0.1)} = 5 - 5 e^{-0.3} \approx 1.30$$

which looks correct according to the following graph of $I(t)$.

![R-L Circuit: current I(t)](image)
[c] At what time does the current equal half the steady-state current?

Solve

\[ I(t) = \frac{5}{2} \]

or

\[ 5 - 5e^{-3t} = 2.5 \]

to get

\[ t = -\frac{1}{3} \ln(0.5) \approx 0.231 \]

This answer could have been approximated by graphing \( I(t) \) on a graphing calculator and zooming or tracing the curve. The preceding graph of \( I(t) \) provides a visual check of this answer.

[d] What is the average current over the first five time units for this circuit?

By definition, the time unit for this R-L DC circuit is

\[ \tau = \frac{L}{R} = \frac{4}{12} = \frac{1}{3} \]

By definition of the average value of a function (see, e.g., p. 473 of Stewart), the average current over first five time units is

\[ I_{avg} = \frac{1}{5/3} \int_{0}^{5/3} (5 - 5e^{-3t}) \, dt = 4 + e^{-5} \approx 4.01 \text{ amps} \]
Details of solution by separation of variables

After multiplying both sides of the ODE
\[ \frac{dI}{dt} = 15 - 3I \]
by \( dt \), we get the ODE in differential form
\[ dI = (15 - 3I) \, dt \]
Divide both sides by \( 15 - 3I \) in order to separate variables: put anything involving \( I \) on one side and anything involving \( t \) on the other side:
\[ \frac{dI}{15 - 3I} = dt \quad (1) \]
Now we are allowed to integrate each side separately and still have equality. The right side of equation (1) is easy:
\[ \int dt = t + C \]
where \( C \) is an arbitrary constant. The left side of equation (1) looks related to the integral \( \int \frac{1}{x} \, dx \). So we use the substitution
\[ x = 15 - 3I \]

to get
\[ \frac{dx}{dI} = -3 \]
or
\[ dI = -\frac{1}{3} \, dx \]
Then in equation (1) we replace \( 15 - 3I \) with \( x \) and \( dI \) with \( -\frac{1}{3} \, dx \) and integrate in order to get the left side to equal
\[ \int \frac{1}{15 - 3I} \, dI = \int \frac{1}{x} \left(-\frac{1}{3} \, dx\right) \]
\[ = -\frac{1}{3} \int \frac{1}{x} \, dx \]
\[ = -\frac{1}{3} \ln |x| + C \]
\[ = -\frac{1}{3} \ln |15 - 3I| + C \]
Hence equation (1), after both sides are integrated, becomes (collecting all arbitrary constants on the right hand side as a single arbitrary constant)
\[ -\frac{1}{3} \ln |15 - 3I| = t + C \quad (2) \]
Since there is no current when the switch is thrown, we let \( I = 0 \) when \( t = 0 \) to solve for \( C \)

\[
\frac{1}{3} \ln |15 - 0| = 0 + C \implies C = -\frac{1}{3} \ln 15
\]

and so equation (2) becomes

\[
\frac{1}{3} \ln |15 - 3I| = -\frac{1}{3} \ln 15 + t
\]

It is usually preferable to solve for the dependent variable, \( I \) in this case. To do that, we first multiply both sides of the last equation by \(-3\) to get

\[
\ln |15 - 3I| = \ln 15 - 3t
\]

then take the exponential (inverse logarithm) of both sides

\[
e^{\ln |15 - 3I|} = e^{\ln 15 - 3t}
\]

and then use a property of exponentials

\[
e^{a+b} = e^a \times e^b
\]

with \( a = \ln 15 \) and \( b = -3t \) to get from equation (3)

\[
|15 - 3I| = e^{\ln 15} \times e^{-3t} = 15 e^{-3t}
\]

since \( e^{\ln 15} = 15 \). Now \( |x| = c \implies x = \pm c \) so we have

\[
15 - 3I = \pm 15 e^{-3t}
\]

Since we know that \( I = 0 \) at \( t = 0 \), we determine the sign to be +, allowing us to solve for \( I \) by dividing both sides of the last equation by 3 and then isolating \( I \) on one side

\[
I(t) = 5 \left(1 - e^{-3t}\right), \quad t \geq 0
\]