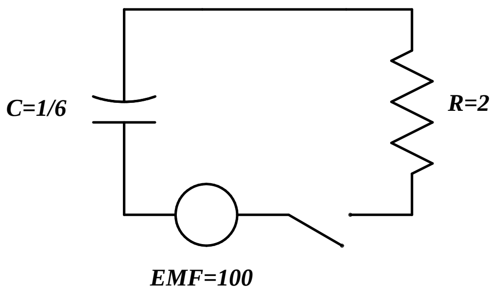
III. Practice Problem 1: R-C DC Circuit

Task: Work on solving for the charge for the given circuit; indicated links give (partial) solutions.



An R-C circuit consists of a 100 volt DC battery connected in series with a 2 ohm resistor and a 1/6 farad capacitor; there is no initial charge on the capacitor and no current flows until the switch is closed. The associated Initial Value Problem is

$$2\,Q'(t) + \frac{Q(t)}{1/6} = 100, \quad Q(0) = 0$$

Method of Separation of Variables:

[a] Put the ODE in differential form.

[b] Use algebra to separate variables.

[c] Integrate both sides of the preceding equation to get a general solution to the ODE.

[d] Use the initial condition to compute the value of the arbitrary constant and determine the implicit solution to the IVP.

[e] Use the rules of logs and exponentials in order to solve the implicit solution to the ODE for the charge $Q(t)$ for this circuit.

[f] Rework the solution when the capacitor in this circuit is charged to 30 coulombs at the time when the switch is closed.

[a] Put the ODE in differential form.

The ODE

$$2\,Q'(t) + \frac{Q(t)}{1/6} = 100$$

becomes

$$Q'(t) + 3\,Q(t) = 50$$

or

$$\frac{dQ}{dt} = 50 - 3\,Q$$

which, after multiplying through by dt, becomes

$$dQ = (50 - 3\,Q)\,dt$$

[b] Use algebra to separate variables.

The differential form

$$dQ = (50 - 3\,Q)\,dt$$

can be written

$$\frac{dQ}{50 - 3\,Q} = dt$$

so that all terms involving Q are on one side of the equation and all terms involving t on the other side.

[c] Integrate both sides of the preceding equation to get a general solution to the ODE.

Integrating both sides of

$$\frac{dQ}{50-3\,Q} = dt$$

gives

$$\int \frac{dQ}{50-3\,Q} = \int dt$$

We use the substitution

$$x = 50 - 3\,Q$$

$$\text{to get} \qquad \frac{dx}{dQ} = -3$$

$$\text{or} \qquad dQ = -\frac{1}{3}dx$$

so that

$$\begin{aligned} \int \frac{1}{50-3\,Q}\,dQ &= \int \frac{1}{x}\left(-\frac{1}{3}dx\right) \\ &= -\frac{1}{3}\int \frac{1}{x}\,dx \\ &= -\frac{1}{3}\ln|x| + C \\ &= -\frac{1}{3}\ln|50-3\,Q| + C \end{aligned}$$

Hence the general solution to the ODE in implicit form is

$$-\frac{1}{3}\ln|50-3\,Q| = t + C$$

[d] Use the initial condition to compute the value of the arbitrary constant and determine the implicit solution to the IVP.

Since $Q = 0$ when $t = 0$, we put $t = 0$ into

$$-\frac{1}{3}\ln|50 - 3\,Q| = t + C$$

to get

$$-\frac{1}{3}\ln|50 - 0| = 0 + C \Longrightarrow C = -\frac{1}{3}\ln 50$$

This leads us to the solution to the IVP in implicit form:

$$-\frac{1}{3}\ln|50 - 3\,Q| = t - \frac{1}{3}\ln 50$$

or, multiplying through by -3,

$$\ln|50 - 3\,Q| = \ln 50 - 3t$$

[e] Use the rules of logs and exponentials in order to solve the implicit solution to the ODE for the charge $Q(t)$ for this circuit.

Take the antilogarithm of both sides of the solution to the IVP

$$\ln|50 - 3\,Q| = \ln 50 - 3t$$

to get

$$e^{\ln|50-3\,Q|} = e^{\ln 50 - 3t}$$

which, when simplified, yields

$$|50 - 3\,Q| = 50\,e^{-3t}$$

or

$$50 - 3\,Q = \pm 50\,e^{-3t}$$

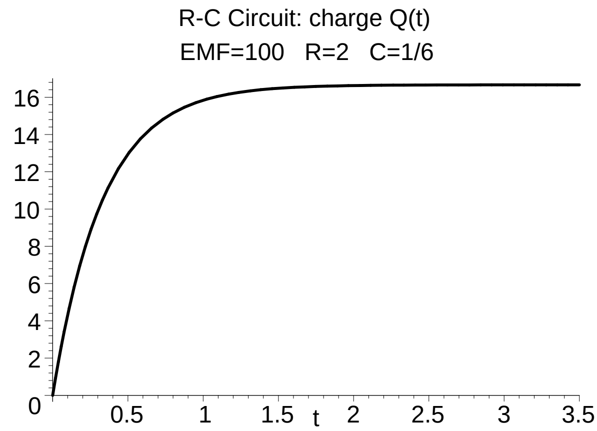
Since $Q = 0$ when $t = 0$, the sign must be $+$, allowing us to solve for Q:

$$50 - 3\,Q = +50\,e^{-3t}$$

or

$$Q(t) = \frac{50}{3}\left(1 - e^{-3t}\right)$$

whose graph is plotted below.



[f] Rework the solution when the capacitor in this circuit is charged to 30 coulombs at the time when the switch is closed.

Start with the general solution

$$-\frac{1}{3}\ln|50 - 3\,Q| = t + C$$

from part [c] and substitute $Q = 30$ and $t = 0$:

$$-\frac{1}{3}\ln|50 - 3(30)| = 0 + C$$

which yields

$$C = -\frac{1}{3}\ln(40)$$

Hence

$$-\frac{1}{3}\ln|50 - 3\,Q| = t - \frac{1}{3}\ln(40)$$

which we can solve as before.

Multiply by -3:

$$\ln|50 - 3\,Q| = -3t + \ln(40)$$

Take antilogs:

$$|50 - 3\,Q| = e^{-3t+\ln(40)}$$

Simplify exponentials:

$$e^{-3t+\ln(40)} = e^{-3t} \times e^{\ln(40)} = e^{-3t} \times 40$$

to arrive at:

$$|50 - 3\,Q| = 40\,e^{-3t}$$

or

$$50 - 3\,Q = \pm 40\,e^{-3t}$$

Now $Q = 30$ when $t = 0$ means

$$50 - 3(30) = \pm 40\,e^{0}$$

or

$$-40 = \pm 40$$

So the sign must be $-$ and the solution to the IVP is

$$50 - 3\,Q = -40\,e^{-3t}$$

or

$$Q(t) = \frac{10}{3}\left(5 + 4e^{-3t}\right)$$

Charge (below left) and current (below right) are plotted for $Q(0) = 0$ (thinner curve) and $Q(0) = 30$ (thicker curve).