III. Practice Problem 2: R-L DC Circuit

Task: Work on solving for the current for the given circuit; indicated links give (partial) solutions.

An R-L circuit consists of a 100 volt DC battery connected in series with a 0.1 henry inductor and a 50 ohm resistor; current flows when the open switch is closed. (This is the example on pp. 168–9 of Hambley: Electrical Engineering—Principles & Applications). The associated Initial Value Problem is

\[ 0.1 I'(t) + 50 I(t) = 100, \quad I(0) = 0 \]

Method of Separation of Variables:

[a] Put the ODE in differential form.
[b] Use algebra to separate variables.
[c] Integrate both sides of the preceding equation to get a general solution to the ODE.
[d] Use the initial condition to compute the value of the arbitrary constant and determine the implicit solution to the IVP.
[e] Use the rules of logs and exponentials in order to solve the implicit solution to the ODE for the current \( I(t) \) for this circuit.
[f] Rework the solution when there is an initial current of 5 amps in this circuit at time \( t = 0 \).
[a] Put the ODE in differential form.

The ODE

\[ 0.1 I'(t) + 50 I(t) = 100 \]

becomes

\[ I'(t) + 500 I(t) = 1000 \]

or

\[ \frac{dI}{dt} = 1000 - 500 I \]

which, after multiplying through by \( dt \), becomes

\[ dI = (1000 - 500 I) \, dt \]
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[b] Use algebra to separate variables.

The differential form

\[ dI = (1000 - 500I) \, dt \]

can be written

\[ \frac{dI}{1000 - 500I} = dt \]

so that all terms involving \( I \) are on one side of the equation and all terms involving \( t \) on the other side.
Integrate both sides of the preceding equation to get a general solution to the ODE.

Integrating both sides of

\[ \frac{dI}{1000 - 500I} = dt \]

gives

\[ \int \frac{dI}{1000 - 500I} = \int dt \]

We use the substitution

\[ x = 1000 - 500I \]

to get

\[ \frac{dx}{dI} = -500 \]

or

\[ dI = -0.002 \, dx \]

so that

\[ \int \frac{1}{1000 - 500I} \, dI = \int \frac{1}{x} (-0.002) \, dx \]

\[ = -0.002 \int \frac{1}{x} \, dx \]

\[ = -0.002 \ln |x| + C \]

\[ = -0.002 \ln |1000 - 500I| + C \]

Hence the general solution to the ODE in implicit form is

\[ -0.002 \ln |1000 - 500I| = t + C \]
[d] Use the initial condition to compute the value of the arbitrary constant and determine the implicit solution to the IVP.

Since \( I = 0 \) when \( t = 0 \), we put \( t = 0 \) into

\[-0.002 \ln |1000 - 500 I| = t + C\]

to get

\[-0.002 \ln |1000 - 0| = 0 + C \implies C = -0.002 \ln 1000\]

This leads us to the solution to the IVP in implicit form:

\[-0.002 \ln |1000 - 500 I| = t - 0.002 \ln 1000\]

or, multiplying through by \(-500\),

\[\ln |1000 - 500 I| = \ln 1000 - 500t\]
[e] Use the rules of logs and exponentials in order to solve the implicit solution to the ODE for the current $I(t)$ for this circuit.

Take the antilogarithm of both sides of the solution to the IVP

$$\ln |1000 - 500 I| = \ln 1000 - 500t$$

to get

$$e^{\ln |1000 - 500 I|} = e^{\ln 1000 - 500t}$$

which, when simplified, yields

$$|1000 - 500 I| = 1000 e^{-500t}$$

or

$$1000 - 500 I = \pm 1000 e^{-500t}$$

Since $I = 0$ when $t = 0$, the sign must be $+$, allowing us to solve for $I$:

$$1000 - 500 I = +1000 e^{-500t}$$

or

$$I(t) = 2 - 2 e^{-500t}$$

whose graph is plotted below.
Rework the solution when there is an initial current of 5 amps in this circuit at time $t = 0$.

Start with the general solution

$$-0.002 \ln |1000 - 500 I| = t + C$$

from part [c] and substitute $I = 5$ and $t = 0$:

$$-0.002 \ln |1000 - 500(5)| = t + C$$

which yields

$$C = -0.002 \ln(1500)$$

Hence

$$-0.002 \ln |1000 - 500(5)| = t - 0.002 \ln(1500)$$

which we can solve as before.

Multiply by $-500$:

$$\ln |1000 - 500 I| = -500t + \ln(1500)$$

Take antilogs:

$$|1000 - 500 I| = e^{-500t+\ln(1500)}$$

Simplify exponentials:

$$e^{-500t+\ln(1000)} = e^{-500t} \times e^{\ln(1500)} = e^{-500t} \times 1500$$

to arrive at:

$$|1000 - 500 I| = 1500 e^{-500t}$$

or

$$1000 - 500 I = \pm 1500 e^{-500t}$$

Now $I = 5$ when $t = 0$ means

$$1000 - 500(5) = \pm 1500 e^{0}$$

or

$$-1500 = \pm 1500$$

So the sign must be $-$ and the solution to the IVP is

$$1000 - 500 I = -1500 e^{-500t}$$

or

$$I(t) = 2 + 3 e^{-500t}$$

Current is plotted below for $I(0) = 0$ (thinner curve) and $I(0) = 5$ (thicker curve).
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R-L Circuit: current $i(t)$
EMF = 100, $R = 50$, $L = 0.1$