II. Example 1: Solving First Order Linear ODE by Integrating Factor

**Task:** Solve the initial value problem (IVP)

\[ 3y' + 15y = 24e^{3x}, \quad y = -2 \quad \text{when} \quad x = 0 \tag{*} \]

**Solution:** Method of Integrating Factor

1. Divide through by 3 to get the ODE in (*) into **standard form** with 1 as the coefficient of \( y' \)

\[ y' + 5y = 8e^{3x} \]

2. Define the **integrating factor** from the coefficient of \( y \) in the preceding equation

\[ \mu(x) = e^{\int 5 \, dx} = e^{5x} \]

where we may take the arbitrary constant of integration to be 0.

3. Multiply both sides of the standard form ODE

\[ y' + 5y = 8e^{3x} \]

by the integrating factor \( e^{5x} \) to get

\[ e^{5x}y' + 5e^{5x}y = 8e^{8x} \]

which can be written (using the product rule for derivatives) as

\[ \left[e^{5x}y\right]' = 8e^{8x} \]

4. Integrate both sides of this last equation to get

\[ e^{5x}y = \int 8e^{8x} \, dx \]

or

\[ e^{5x}y = e^{8x} + C \]

and divide through by \( e^{5x} \) to get the solution to the ODE in (*) as

\[ y = e^{3x} + Ce^{-5x} \]

using the law of exponents \( e^{-5x} \times e^{8x} = e^{3x} \).

5. To satisfy the intial condition \( y(0) = -2 \) in (*) we put \( y = -2 \) and \( x = 0 \) into

\[ y = e^{3x} + Ce^{-5x} \]

resulting in

\[ -2 = e^{3\cdot0} + Ce^{5\cdot0} \]

\[ \Rightarrow -2 = 1 + C \cdot 1 \]

\[ \Rightarrow -2 = 1 + C \]

\[ \Rightarrow -3 = C \]

So the solution to the IVP (*) is

\[ y = e^{3x} - 3e^{-5x} \]