II. Example 1: First Order Separable ODEs

Task: Solve the initial value problem (IVP)

\[ y' = \frac{3e^{3x}}{y + 5}, \quad y = 1 \quad \text{when} \quad x = 0 \]  

(\ast)

Solution: Method of Separation of Variables

1. Multiply both sides of the ODE in (\ast) by \( y + 5 \) to get it in the form

\[ (y + 5)y' = 3e^{3x} \]

2. Replace \( y' \) by (the Leibniz) differential form \( \frac{dy}{dx} \)

\[ (y + 5)\frac{dy}{dx} = 3e^{3x} \]

3. Multiply both sides of this last ODE by \( dx \):

\[ (y + 5)dy = 3e^{3x}dx \]

thus “separating variables”: each side of the equation now has one and only one variable present.

4. Integrate both sides

\[ \int (y + 5)dy = \int 3e^{3x}dx \]

5. Evaluate both indefinite integrals and put all arbitrary constants on the right hand side

\[ \frac{1}{2}y^2 + 5y = e^{3x} + C \]

6. To solve for \( y \), multiply both sides of the preceding equation by 2

\[ y^2 + 10y = 2e^{3x} + C \]

and complete the square (or use the quadratic equation) to get

\[ (y + 5)^2 - 25 = 2e^{3x} + C \]
or
\[(y + 5)^2 = 2e^{3x} + C\]

where 25 gets absorbed into the arbitrary constant \(C\). Taking the square root of both sides of the preceding equation

\[y + 5 = \sqrt{2e^{3x} + C}\]

which leads to

\[y = \sqrt{2e^{3x} + C} - 5\]

as the general solution to the ODE in (\(*\)). To satisfy the initial condition \(y(0) = 1\) in (\(*\)) we need to have

\[
\begin{align*}
\sqrt{2e^{3\cdot0}} + C - 5 &= 1 \\
\implies \sqrt{2} \cdot 1 + C &= 6 \\
\implies 2 + C &= 36 \\
\implies C &= 34
\end{align*}
\]

So the solution to the IVP (\(*\)) is

\[y = \sqrt{2e^{3x} + 34} - 5\]