II. Solving First Order Separable ODEs

An ordinary differential equation of the form

\[ y' = \frac{p(x)}{q(y)} \]

is called a first order separable ODE where \( p(x) \) is a function of only the variable \( x \) and \( q(y) \neq 0 \) is a function of only the variable \( y \).

Solution: Method of Separation of Variables

1. Multiply both sides of the ODE by \( q(y) \) to get it in the form

\[ q(y) y' = p(x) \]

2. Replace \( y' \) by (the Leibniz) differential form \( \frac{dy}{dx} \)

\[ q(y) \frac{dy}{dx} = p(x) \]

3. Multiply both sides of this last ODE by \( dx \):

\[ q(y) dy = p(x) dx \]

thus “separating variables”: each side of the equation now has one and only one variable present.

4. Integrate both sides (an allowable operation because of the Chair Rule for derivatives)

\[ \int q(y) dy = \int p(x) dx \]

5. If both integrals can be evaluated, then we have an “implicit” solution to the original ODE in the form

\[ Q(y) = P(x) + C \]

after collecting the integrals’ arbitrary constants on one side as \( C \).

6. You might be able to solve that last equation for \( y \) in terms of \( x \)

\[ y = F(x) \]

or perhaps \( x \) in terms of \( y \)

\[ x = G(y) \]
in order to get an “explicit” solution to the original ODE.

**Special case:** *Linear first order ODE with constant coefficients*

To solve

\[ ay' + by = c \]

where \( a \neq 0, b \neq 0, \) and \( c \) are constants.

1. Divide through by the number \( a \) to get an ODE of the form

\[ y' + py = q \]

where numbers \( p = b/a \) and \( q = c/a \).

2. Subtract \( py \) from both sides and write \( y' \) in differential notation \( \frac{dy}{dx} \):

\[ \frac{dy}{dx} = q - py \]

3. Divide both sides of the ODE by \( q - py \), multiply both sides by \( dx \)

\[ \frac{dy}{q - py} = dx \]

and then integrate

\[ \int \frac{dy}{q - py} = \int dx \]

4. Use the method of substitution with \( u = q - py \) so that \( du = -p \, dy \) or \( dy = (-1/p) \, du \) to evaluate the left hand side of the preceding equation

\[ \int \frac{dy}{q - py} = \int \frac{(-1/p) \, du}{u} \]

\[ = \frac{-1}{p} \ln |u| + C_1 \]

\[ = \frac{-1}{p} \ln |q - py| + C_1 \]

where \( C_1 \) is an arbitrary constant.

5. Putting parts 3. and 4. together

\[ \frac{-1}{p} \ln |q - py| + C_1 = x + C_2 \]
or, after combining the two arbitrary constants on the right side

$$\ln |q - py| = -px + C$$

which, after taking the antilogarithm of both sides, yields

$$|q - py| = e^{-px+C} = e^{-px}e^C = Ae^{-px}$$

where $A = e^C$ is an arbitrary positive constant. Then after eliminating the absolute values we are left with

$$q - py = \pm Ae^{-px}$$

from which we can solve for $y$

$$y = \frac{q}{p} - \frac{\pm A}{p}e^{-px}$$

We replace $\frac{\pm A}{p}$ by $C$ to stand for an arbitrary constant. And looking back we see that since $q = c/a$ and $p = b/a$, the solution to the ODE may be written

$$y = \frac{c}{b} + Ce^{-\frac{b}{a}x}$$