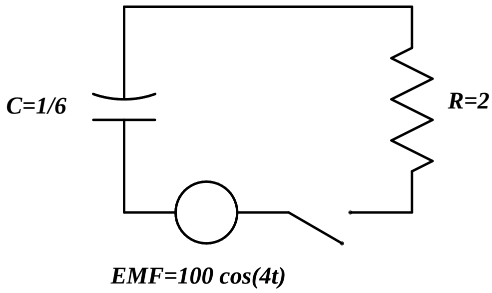
I. Example 1: R-C AC Circuit

Note: a circuit with oscillating or periodic EMF is called an *alternating current circuit* or *AC circuit*.

Questions:

[a] Use Kirchhoff's law to write the Initial Value Problem — ODE and initial condition(s) — for the simple circuit consisting of a $100\cos(4t)$ volt AC generator connected in series with a 2 ohm resistor and a $\frac{1}{6}$ farad capacitor. There is no charge on the capacitor is 0 when the open switch is closed.



[b] Verify that $Q(t) = 6\cos(4t) + 8\sin(4t) - 6\,e^{-3t}$ is the solution to this IVP.

[c] Find $I(t)$, the current at time t, then graph both charge $Q(t)$ and current $I(t)$.

Answers:

[a] By Kirchhoff's law we have that $E_R + E_C = E$ which, with $E_R = R \cdot Q'(t)$ and $E_C = Q(t)/C$, translates into the Initial Value Problem (for $t \geq 0$)

$$2\,Q'(t) + \frac{1}{1/6}Q(t) = 100\cos(4t)\,, \qquad Q(t) = 0 \quad \text{at} \quad t = 0$$

or

$$Q'(t) + 3\,Q(t) = 50\cos(4t)\,, \qquad Q(t) = 0 \quad \text{at} \quad t = 0 \tag{$*$}$$

[b] If $Q(t) = 6\cos(4t) + 8\sin(4t) - 6\,e^{-3t}$ then its derivative

$$\begin{aligned} Q'(t) &= 6\,(-4\sin(4t)) + 8\,(4\cos(4t)) - 6\left(-3\,e^{-3t}\right) \\ &= -24\sin(4t) + 32\cos(4t) + 18\,e^{-3t} \end{aligned}$$

and so the left hand side of the ODE in $(*)$ above becomes

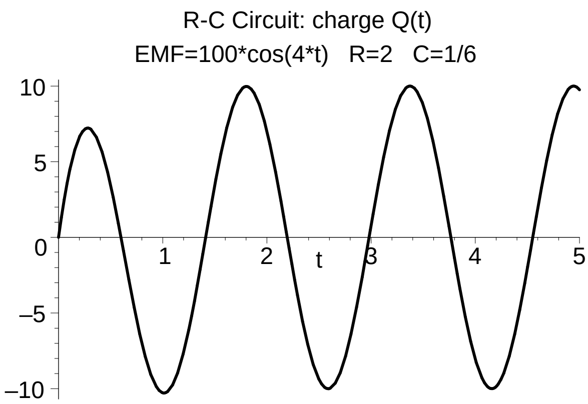
$$\begin{aligned} Q'(t) + 3\,Q(t) &= -24\sin(4t) + 32\cos(4t) + 18\,e^{-3t} + 3\left(6\cos(4t) + 8\sin(4t) - 6\,e^{-3t}\right) \\ &= -24\sin(4t) + 32\cos(4t) + 18\,e^{-3t} + 18\cos(4t) + 24\sin(4t) - 18\,e^{-3t} \\ &= 50\cos(4t) \end{aligned}$$

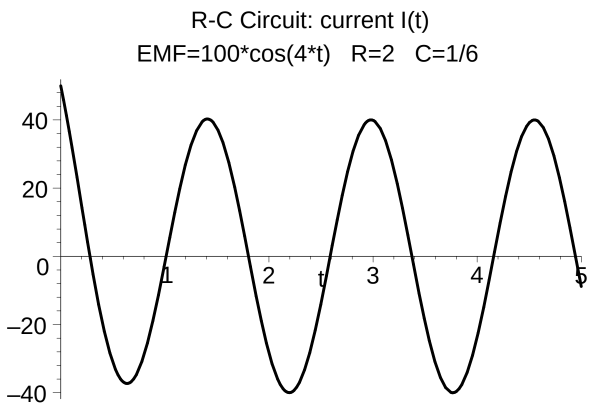
Hence $Q(t)$ satisfies the ODE in $(*)$. Also, $Q(0) = 6\cos(0) + 8\sin(0) - 6\,e^0 = 6 + 0 - 6 = 0$ and $Q(t)$ also satisfies the IC in $(*)$.

[c] Current is the time derivative of charge:

$$I(t) = Q'(t) = -24\sin(4t) + 32\cos(4t) + 18e^{-3t}\,, \qquad t > 0$$

Graphs of $Q(t)$ and $I(t)$ are below.





Notice that $Q(t)$ settles into a *steady-state* $6\cos(4t) + 8\sin(4t)$ after the *transient term* $6\,e^{-3t}$ "dies off" (converges to 0). Can you identify the *steady-state term* and the *transient term* for $I(t)$?