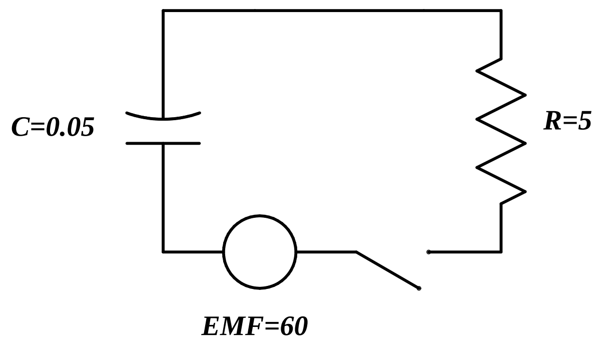
I. Example 2: R-C DC Circuit

Questions:

[a] Use Kirchhoff's law to write the Initial Value Problem — ODE and initial condition(s) — for the simple circuit consisting of a 60 volt DC battery connected in series with a 0.05 farad capacitor and a 5 ohm resistor. There is no charge on the capacitor and current flows when the open switch is closed. (Note: This is Exercise #27 on p. 521 and p. 528 of Stewart: **Calculus—Concepts and Contexts**, 2nd ed.)



[b] Verify that $Q(t) = 3\left(1 - e^{-4t}\right)$, $t \geq 0$ is the solution to the IVP in part [a].

[c] Find $I(t)$, the current at time t, then graph both charge $Q(t)$ and current $I(t)$.

[d] What is the asymptotic limit of $Q(t)$ as $t \to \infty$? This is called the *steady-state* charge and we will label it Q_∞.

[e] At what time t does the charge $Q(t)$ reach 50% of its steady state value?

Answers:

[a] By Kirchhoff's law we have that $E_R + E_C = E$ which, with $E_R = R \cdot Q'(t)$ and $E_C = Q/C$, translates into the Initial Value Problem (for $t \geq 0$)

$$5\,Q'(t) + \frac{Q(t)}{0.05} = 60\,, \qquad Q(t) = 0 \quad \text{at} \quad t = 0$$

or, after simplifying,

$$Q'(t) + 4\,Q(t) = 12\,, \qquad Q(t) = 0 \quad \text{at} \quad t = 0 \qquad (*)$$

[b] If $Q(t) = 3\left(1 - e^{-4t}\right)$ then its derivative

$$Q'(t) = 3\left(0 - (-4)\,e^{-4t}\right) = 3\left(4\,e^{-4t}\right) = 12e^{-4t}$$

and so the left hand side of the ODE in $(*)$ above becomes

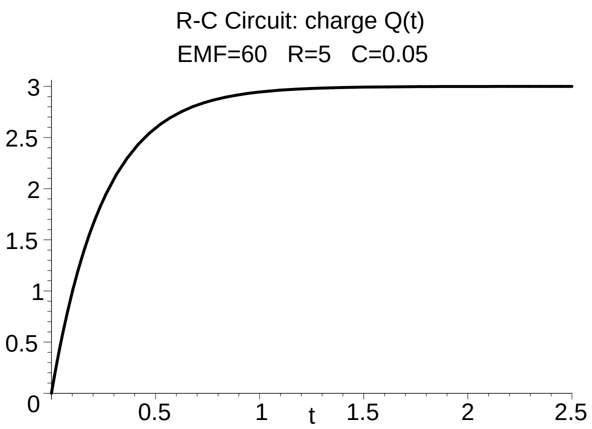
$$\begin{aligned} Q'(t) + 4\,Q(t) &= \left(12e^{-4t}\right) + 4\left(3\left(1 - e^{-4t}\right)\right) \\ &= 12\,e^{-4t} + 12 - 12\,e^{-4t} \\ &= 12 \end{aligned}$$

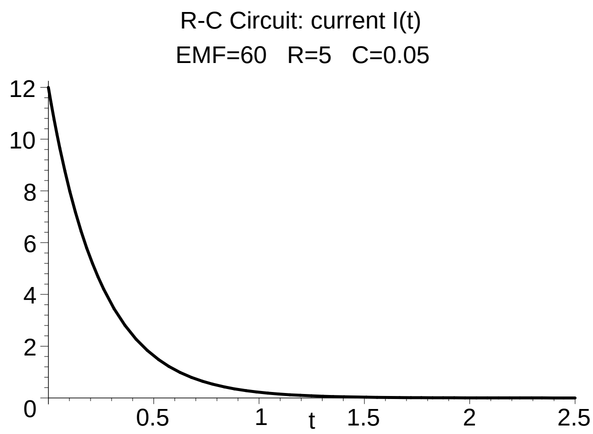
Hence $Q(t)$ satisfies the ODE of $(*)$. Also, $Q(0) = 3\left(1 - e^0\right) = 3(1-1) = 0$ and $Q(t)$ also satisfies the IC of $(*)$.

[c] Current is the time derivative of charge:

$$I(t) = Q'(t) = 12e^{-4t} \quad \text{for} \quad t > 0$$

Graphs of $Q(t)$ and $I(t)$ are below.





[d] As $t \to \infty$ we have $Q(t) = 3\left(1 - e^{-4t}\right) \to 3\,(1-0) = 3$ as indicated in the preceding graph of $Q(t)$. The steady-state charge $Q_\infty = 3$ coulombs.

We can see that as $t \to \infty$ and the graph of $Q(t)$ levels off toward the steady-state value Q_∞, then the slope of the graph $Q'(t) \to 0$. If we take the limit $\lim_{t\to\infty}$ of both sides of the ODE in $(*)$ we get

$$\lim_{t\to\infty} \left(Q'(t) + 4\,Q(t)\right) = \lim_{t\to\infty} (12)$$

which yields $0 + 4\,Q_\infty = 12$ or $Q_\infty = 12/4 = 3$.

Remark. An R-C circuit with constant DC voltage E has steady-state charge $Q_\infty = E \cdot C$.

[d] We need to solve $Q(t) = 0.50 I_\infty$. In this circuit, this is the same as $3\left(1 - e^{-4t}\right) = 0.5(3)$ which, after canceling the threes, leads to

$$1 - e^{-4t} = 0.5$$

$$\Longrightarrow e^{-4t} = 0.5$$

$$\Longrightarrow -4t = \ln(0.5)$$

$$\Longrightarrow t = -\frac{\ln(0.5)}{4} \approx 0.173$$

This answer could also have been approximated by plotting $Q(t)$ on a graphing calculator and tracing or zooming in on the curve to see where $Q(t)$ achieves the value 1.5 coulombs.

Remark. In R-C DC circuits, a *time unit* τ is defined by $\tau = C \cdot R$. After 5 time units, the charge will be at a little more than 99% of its steady-state: $Q(5\tau) \approx 0.9933\, Q_\infty$.

In this example, $\tau = C \cdot R = 0.05 \cdot 5 = 0.25$ and $5\tau = 1.25$ seconds. We see in the preceding graph of $Q(t) = 3\left(1 - e^{-4t}\right)$ that the charge is indeed very near Q_∞ when t is past 1.25 seconds. Algebraically,

$$Q\,(1.25) = 3\left(1 - e^{-4(1.25)}\right) = 3\left(1 - e^{-5}\right) \approx 3(0.99326) \approx 2.980$$