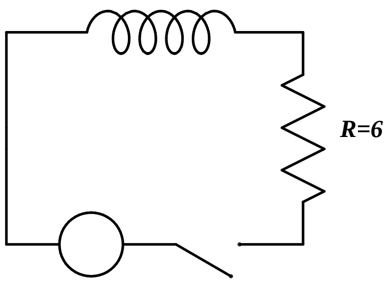
# I. Example 2: R-L AC Circuit

## Questions:

[a] Use Kirchhoff's law to write the Initial value Problem — ODE and initial condition(s) — for the simple circuit consisting of a $100\sin(4t)$ volt AC generator connected in series with a 6 ohm resistor and a 2 henry inductor. Current starts to flow when the open switch is closed.




[b] Verify that $I(t) = -8\cos(4t) + 6\sin(4t) + 8\,e^{-3t}$ is the solution to this IVP and graph $I(t)$.

[c] Compute the root-mean-square generator voltage.

[d] Compute the root-mean-square steady-state current for this circuit.

## Answers:

[a] By Kirchhoff's law we have that $E_L + E_R = E$ which, with $E_L = L \cdot I'(t)$ and $E_R = R \cdot I(t)$, translates into the Initial Value Problem (for $t \geq 0$)

$$2\,\frac{dI}{dt} + 6\,I = 100\,\sin(4t)\,, \qquad I(t) = 0 \quad \text{at} \quad t = 0$$

which simplifies to

$$I'(t) + 3\,I(t) = 50\,\sin(4t)\,, \qquad I(t) = 0 \quad \text{at} \quad t = 0 \qquad (*)$$

[b] If $I(t) = -8\cos(4t) + 6\sin(4t) + 8\,e^{-3t}$ then its derivative

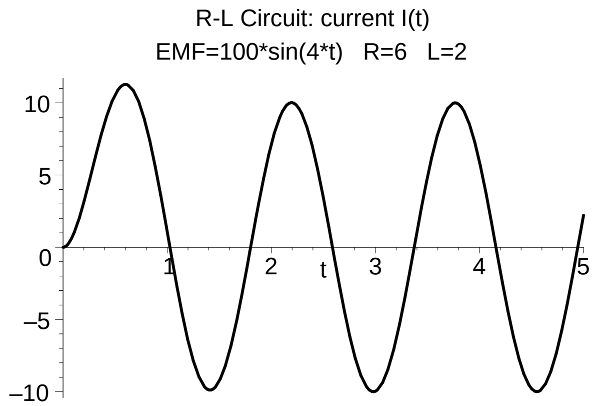
$$\begin{aligned} I'(t) &= -8\big(-4\sin(4t)\big) + 6\big(4\cos(4t)\big) + 8\big(-3\,e^{-3t}\big) \\ &= 32\sin(4t) + 24\cos(4t) - 24\,e^{-3t} \end{aligned}$$

and so the left hand side of the ODE in $(*)$ above becomes

$$\begin{aligned} I'(t) + 3\,I(t) &= 32\sin(4t) + 24\cos(4t) - 24\,e^{-3t} \\ &\qquad + 3\left(-8\cos(4t) + 6\sin(4t) + 8\,e^{-3t}\right) \\ &= 32\sin(4t) + 24\cos(4t) - 24\,e^{-3t} - 24\cos(4t) + 18\sin(4t) + 24\,e^{-3t} \\ &= 50\sin(4t) \end{aligned}$$

Hence $I(t)$ satisfies the ODE in $(*)$. Also, $I(0) = -8\cos(0) + 6\sin(0) + 8\,e^0 = -8+0+8 = 0$ and $I(t)$ also satisfies the IC in $(*)$.

The graph of $I(t)$ follows.




Notice that $I(t)$ settles into a *steady-state* $-8\cos(4t) + 6\sin(4t)$ after the *transient term* $8\,e^{-3t}$ "dies off" (converges to $0$).

[c] The *frequency* $f$ of an oscillatory function is the number of cycles completed in one time unit. If $E(t) = B\,\sin(\omega\,t)$ is the oscillatory function, then the formula for $f$ is $f = \omega/(2\pi)$. The *period* $T = 1/f = 2\pi/\omega$ is the time to complete one cycle.

The *root-mean-square* of an oscillatory function is the square root of the average value over one period of the square of the function.

Putting this into the mathematics of calculus, the root-mean-square of $E(t)$ is defined by

$$E_{rms} = \sqrt{\frac{1}{T}\int_0^T E(t)^2\,dt}$$

If generator voltage $E(t) = 100\,\sin(4t)$, then $T = 2\pi/4 = \pi/2$ and the root-mean-square EMF voltage equals

$$\sqrt{\frac{2}{\pi}\int_0^{\pi/2}\big(100\sin(4t)\big)^2\,dt} = \frac{100}{\sqrt{2}} \approx 70.71 \text{ volts}$$

which turns out to be the "peak voltage" $B = 100$ divided by $\sqrt{2}$.

Root-mean-square voltage is also referred to as the "effective voltage" and is used to compare AC with DC, to compute electrical power, etc. In the U.S. the 120 volt AC household current refers to the root-mean-square voltage and not the peak voltage which equals $120\sqrt{2} \approx 169.7$ V. Can you determine the peak voltage for a system that has an effective 230 AC voltage?

[d] Similarly, the root-mean-square steady-state current in this circuit equals

$$I_{rms} = \sqrt{\frac{2}{\pi}\int_0^{\pi/2} \big(-8\cos(4t) + 6\sin(4t)\big)^2 \, dt} = \frac{10}{\sqrt{2}} \approx 7.07 \text{ amps}$$