SM121 Test II (no calculators) 4 OCT 2010

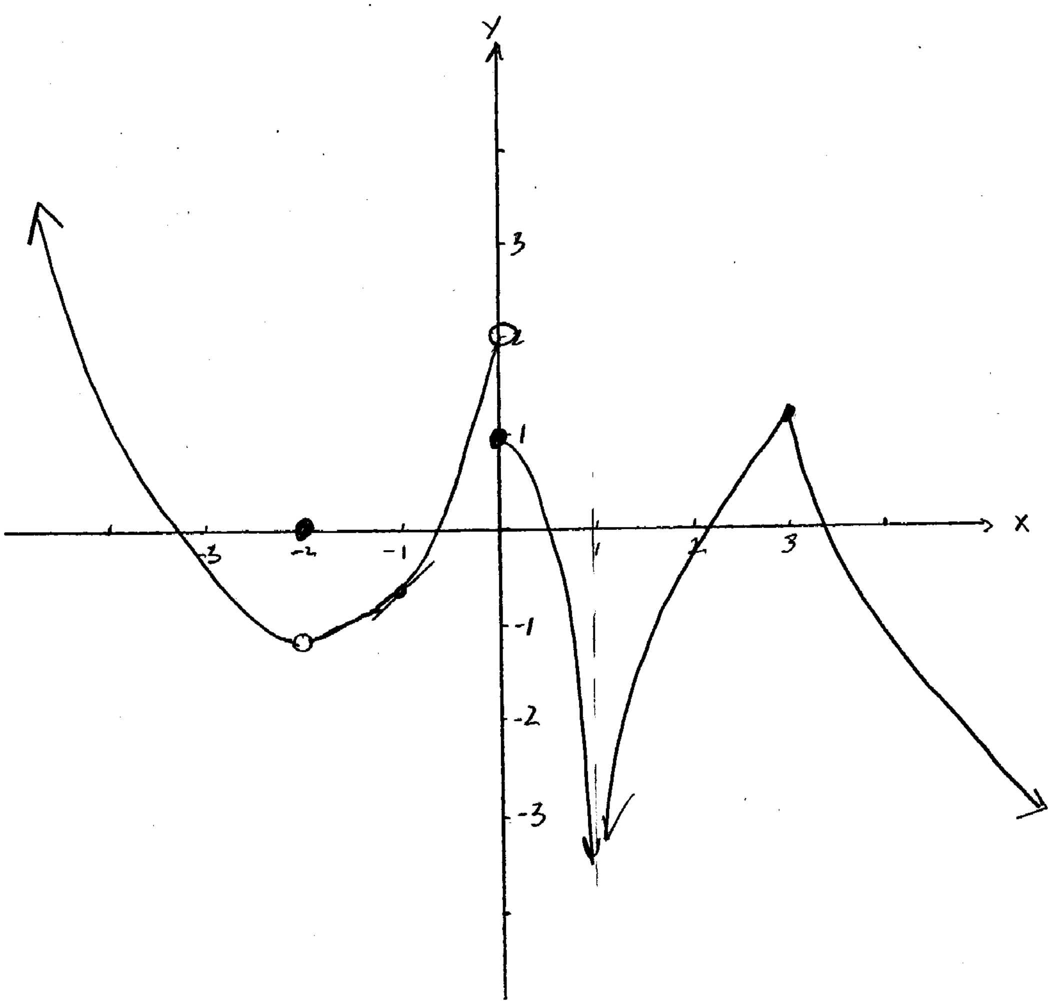
Name KEY Prof. J. D'Archangelo

1. Sketch the graph of a single function which satisfies all of the following:

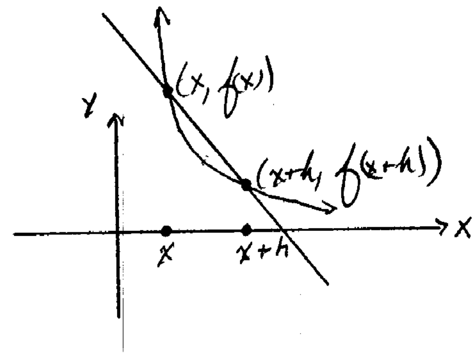
(a) $\lim_{x\to-\infty} f(x)=\infty$; (b) $\lim_{x\to-2} f(x)=-1$; (c) $f(-2)=0$;

(d) $f'(-1)=1$; (e) $\lim_{x\to 0^-} f(x)=2$; (f) f is continuous from the right, but not from the left; at $x=0$.

(g) $\lim_{x\to 1} f(x)=-\infty$; (h) f is continuous at x = 3, but not differentiable there.



2. (a) Label on the graph to the right, the pairs $(x, f(x))$ *and* $(x+h, f(x+h))$.



(b) What does $\dfrac{f(x+h)-f(x)}{h}$ measure ?

Show it on your graph.

The slope of the secant line through $(x, f(x))$ and $(x+h, f(x+h))$ or the average rate of change of f with respect to x over the interval $[x, x+h]$.

(c) Give the definition for $f'(x)$.

$$f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$$

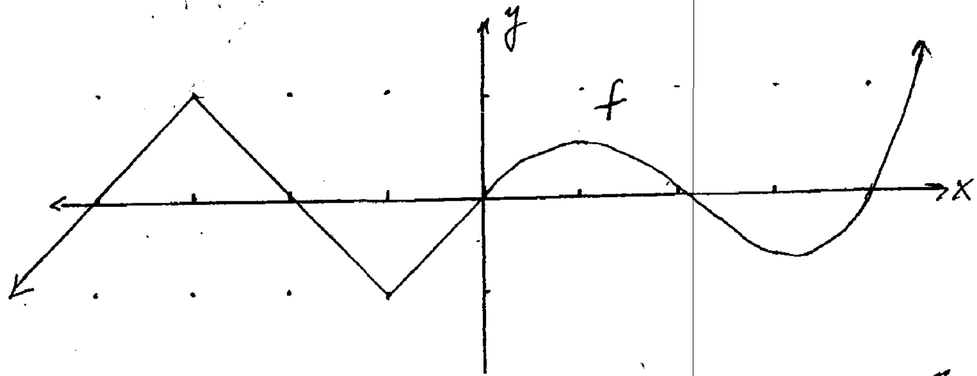
(d) Use your definition in (c) to find $f'(x)$ for $f(x) = \dfrac{1}{x^2}$.

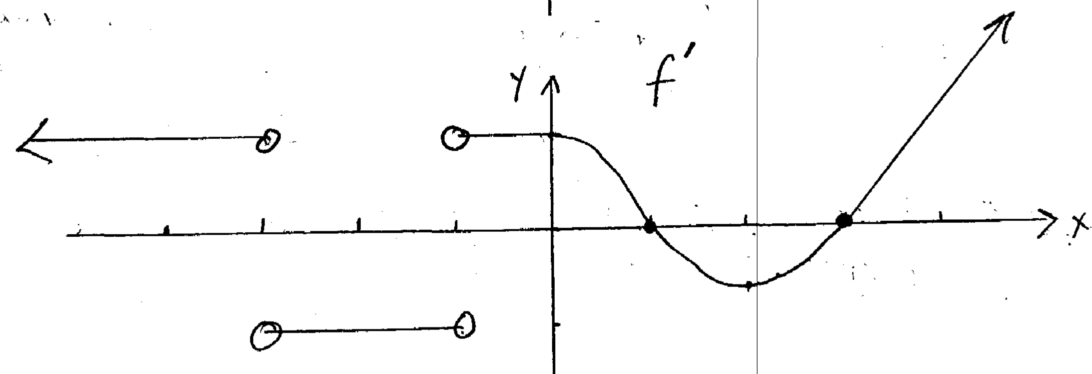
$$f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right] = \lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x^2 - (x+h)^2}{(x+h)^2 x^2}}{h} = \lim_{h \to 0} \frac{\frac{\cancel{x^2} - (\cancel{x^2} + 2xh + h^2)}{(x+h)^2 x^2}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{\cancel{h}(-2x - h)}{(x+h)^2 x^2}}{\cancel{h}} = \frac{-2x}{x^2 \cdot x^2} = \frac{-2}{x^3}$$

3. The graph of a function f is shown. Sketch the graph of f' below it.





4. The purchase price P for a house enclosing x square feet is given by the following table:

x (ft^2)	2,000	2,500	3,000	3,500
P (dollars)	200,000	260,000	310,000	350,000

a) Find the average rate of change of the price with respect to the square footage over the interval [2,000 , 3,000]. Use the correct units.

$$\frac{P(3000) - P(2000)}{3000 - 2000} \frac{\text{dollars}}{ft^2} = \frac{310,000 - 200,000}{1,000} \$/ft^2 = \frac{110,000}{1,000} = 110 \$/ft^2$$

b) Approximate the instantaneous rate of change of the price at x = 3,000 ft^2.

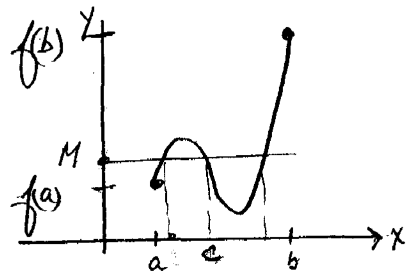
$$P'(3,000) \doteq \frac{P(3,500) - P(2,500)}{3,500 - 2,500} = \frac{350,000 - 260,000}{1,000} = \frac{90,000}{1,000} = 90 \$/ft^2$$

c) Use your answer to part b) to approximate the price of a 3,100 square foot house.

$$P(3,100) = P(3,000)\text{ dollars} + \left(90 \frac{\text{dollars}}{ft^2}\right)(100\, ft^2)$$

$$= 310,000 + 9,000 = 319,000 \text{ dollars}$$

5. (a) State the Intermediate Value Theorem and sketch an illustrative graph.



If f is cont on $[a,b]$ and if M is any number between $f(a)$ and $f(b)$, then there is at least one number c between a and b where $f(c) = M$.

(b) Prove that there is at least one number x between 0 and $\frac{\pi}{2}$ where $\cos(x) - x = 0$.

Let $f(x) = \cos(x) - x$. f is continuous everywhere because $\cos(x)$ and x are.

Let $[a,b] = [0, \pi/2]$

$f(a) = f(0) = \cos(0) - 0 = 1$

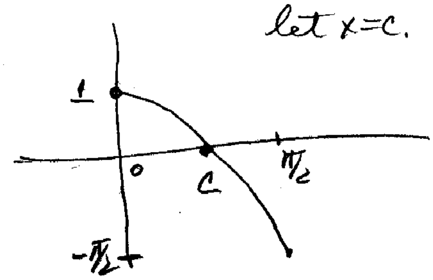
$f(b) = f(\pi/2) = \cos(\pi/2) - \pi/2 = 0 - \pi/2 = -\pi/2$

Let $M = 0$. Since 0 is between $-\pi/2$ and 1

by the IVT there is at least one c in $[0, \pi/2]$ where $f(c) = 0$

or $\cos(c) - c = 0$

let $x = c$.



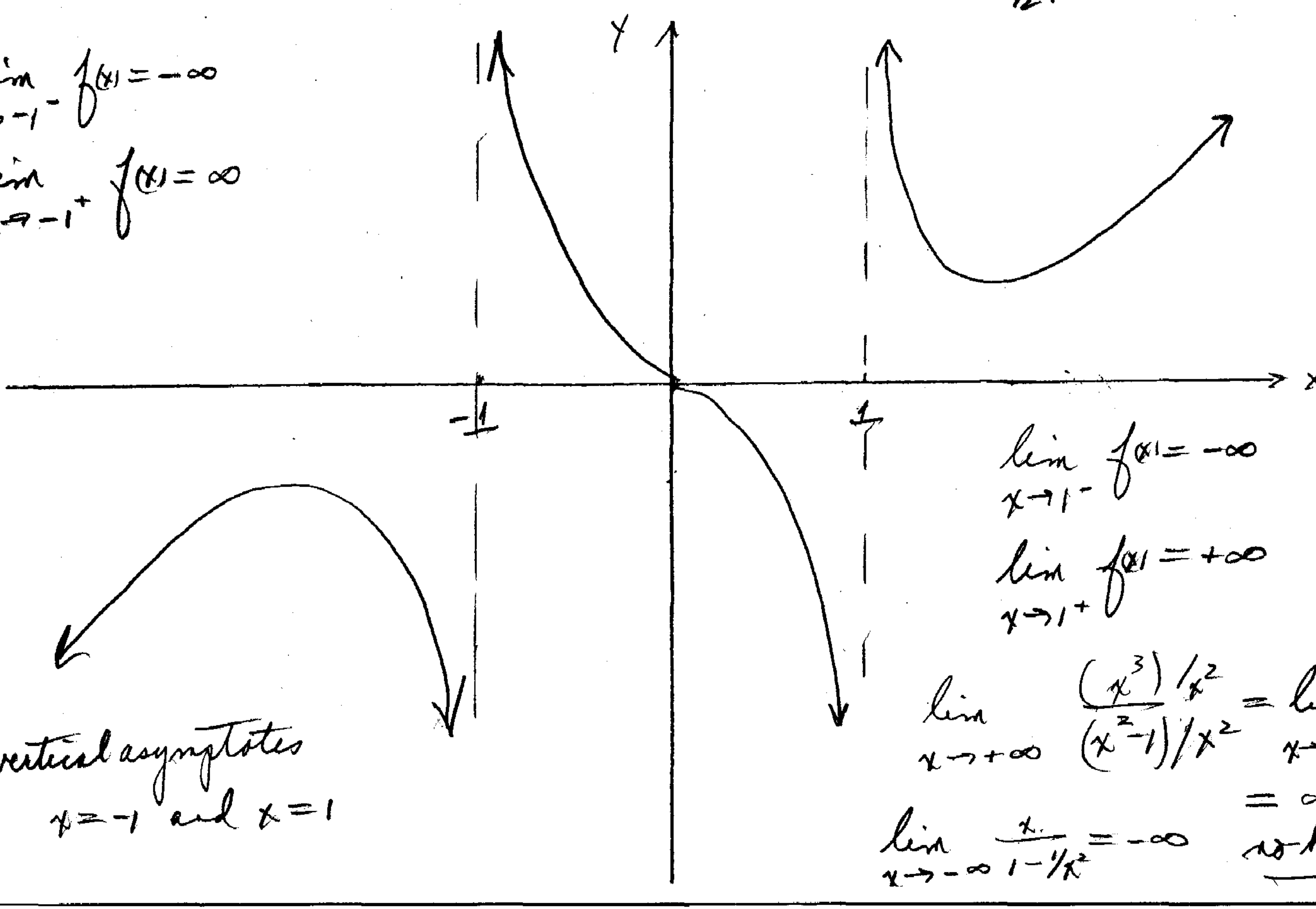
6. Let $f(x) = \dfrac{x^3}{(x-1)(x+1)}$.

(a) Sketch the graph of $y = f(x)$.

(b) What are the horizontal and vertical asymptotes?

$\lim_{x \to -1^-} f(x) = -\infty$

$\lim_{x \to -1^+} f(x) = \infty$



$\lim_{x \to 1^-} f(x) = -\infty$

$\lim_{x \to 1^+} f(x) = +\infty$

$\lim_{x \to +\infty} \dfrac{(x^3)/x^2}{(x^2-1)/x^2} = \lim_{x \to +\infty} \dfrac{x}{1 - 1/x^2} = \infty$

vertical asymptotes

$x = -1$ and $x = 1$

$\lim_{x \to -\infty} \dfrac{x}{1 - 1/x^2} = -\infty$ no horiz asymp