SM 121 Final Exam Answer Key

Tuesday, 12 December 2017, 0755-1045

Identification (fill in right away)

Name: Vrej Zarikian

Alpha: ________________________

Section: ______________

Instructor: ___________________________

Instructions (read carefully)

1. Write your name and alpha on the exam, as well as on the scantron form provided. Also, bubble in your alpha on the scantron form using a #2 pencil.
2. The exam has two parts, each worth 50%. Part I has 20 multiple-choice problems. Record your answers on the scantron form, using a #2 pencil. There is no penalty for guessing. Part II has 10 free-response problems. Record your answers, including all your work, on the exam itself.
3. Make sure that your exam is complete, with 30 problems total.
4. You may use your TI-36X calculator throughout the exam. No other resources are allowed (no books, no notes, no neighbor, no cell-phone, etc.).
5. If you need additional scratch paper, ask your instructor.
6. If you need to use the head, ask your instructor for permission. Leave your cell-phone in the exam room, and place your exam materials face down on your desk.

Part I: Multiple Choice (circle answers & record on scantron)

Question 1. *What are the domain and range for the function*

$$f(x) = 1 + \sqrt{x-2}?$$

(a) *Domain:* $[0, \infty)$, *Range:* $[0, \infty)$

(b) *Domain:* $[2, \infty)$, *Range:* $[0, \infty)$

(c) *Domain:* $[-2, \infty)$, *Range:* $[1, \infty)$

$\boxed{(d)}$ *Domain:* $[2, \infty)$, *Range:* $[1, \infty)$

Question 2. *Suppose that the parabola*

$$y = ax^2 + bx + c$$

passes through the points $(0, 10)$, $(1, 7)$, *and* $(2, 0)$. *Then which of the following statements about the leading coefficient is true?*

$\boxed{(a)}$ $a = -2$

(b) $a = -1$

(c) $a = -1/2$

(d) $a = 2$

Question 3. *If* $0 < \theta < \pi/2$ *and* $\sin(\theta) = 3/8$, *then which of the following statements about* $\sec(\theta)$ *is true?*

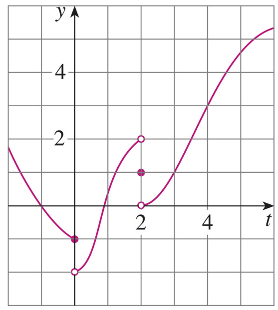
(a) $\sec(\theta) = 8/3$

(b) $\sec(\theta) = 3/\sqrt{55}$

$\boxed{(c)}$ $\sec(\theta) = 8/\sqrt{55}$

(d) $\sec(\theta)$ *cannot be determined from the given information.*

Question 4. *Which of the following statements is true for the function f whose graph is shown below?*



(a) $\lim_{x\to 2} f(x) = 1$;

(b) $f(2)$ does not exist;

[(c)] $\lim_{x\to 2} f(x)$ does not exist;

(d) f is continuous at $x = 2$.

Question 5. *If $f(x) = x^4$, then*

$$\lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$$

(a) equals 81;

(b) equals 0;

[(c)] equals 108;

(d) is undefined.

Question 6. *Suppose that $f(x)$ is continuous on the closed interval $[1,5]$ and differentiable on the open interval $(1,5)$. Suppose further that $f(1) = 3$ and $f(5) = 11$. Which of the following statements follows from the Mean Value Theorem:*

[(a)] for some $1 < c < 5$, $f'(c) = 2$;

(b) for some $1 < c < 5$, $f'(c) = 7$;

(c) for some $1 < c < 5$, $f(c) = 2$;

(d) for some $1 < c < 5$, $f(c) = 7$.

Question 7. *The tangent line to the graph of* $y = \tan(x)$ *at* $x = \pi/4$ *is given by:*

[(a)] $y - 1 = 2(x - \pi/4)$;

(b) $y - 1 = \sec^2(x)(x - \pi/4)$;

(c) $y - \frac{\sqrt{2}}{2} = 2(x - \pi/4)$;

(d) $y - 1 = \frac{1}{2}(x - \pi/4)$.

Question 8. *The absolute maximum value of*

$$f(x) = e^x \cos(x)$$

on the interval $[0, \pi]$ *is closest to*

(a) 1

[(b)] 1.5

(c) 2

(d) 2.5

Question 9. *The area of a triangle is given by the formula* $A = \frac{1}{2}bh$, *where* b *is the length of the base and* h *is the height. If* b *is increasing at a rate of* 3 *cm/s and* h *is increasing at a rate of* 1 *cm/s, how fast is* A *increasing when* $b = 42$ *cm and* $h = 18$ *cm?*

(a) 1.5 cm^2/s

[(b)] 48 cm^2/s

(c) 49.5 cm^2/s

(d) 378 cm^2/s

Question 10. *For* $y = x^{1/x}$, *use logarithmic differentiation to compute* y'.

(a) $y' = \frac{1}{x} \cdot x^{1/x-1}$

(b) $y' = x^{1/x} \ln(x) \cdot -\frac{1}{x^2}$

(c) $y' = \frac{1-\ln(x)}{x^2}$

[(d)] $y' = x^{1/x} \cdot \frac{1-\ln(x)}{x^2}$

Question 11. *Suppose that*

$$f'(x) = \frac{1-x^2}{(1+x^2)^2} \text{ and } f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}.$$

Using this information, find and classify the local extrema of $f(x)$.

(a) $x = -1$ *is a local maximum, and* $x = 1$ *is a local minimum;*

\boxed{(b)} $x = -1$ *is a local minimum, and* $x = 1$ *is a local maximum;*

(c) *both* $x = -1$ *and* $x = 1$ *are local maxima;*

(d) *both* $x = -1$ *and* $x = 1$ *are local minima.*

Question 12. *Which of the following functions is an antiderivative for* $f(x) = \ln(x)$*?*

(a) $F(x) = 1/x$

(b) $F(x) = e^x$

\boxed{(c)} $F(x) = x\ln(x) - x$

(d) $F(x) = \ln(x^2/2)$

Question 13. *Compute the following limit:*

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3}.$$

\boxed{(a)} $-1/6$

(b) 0

(c) 1

(d) *the limit does not exist*

Question 14. *If* $y = \ln(\cos(x^2))$*, then* y' *equals*

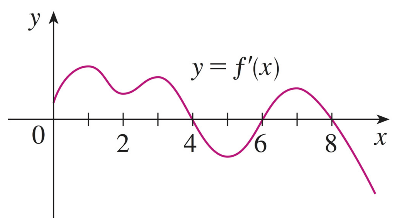
(a) $\frac{1}{\cos(x^2)}$

(b) $\frac{1}{x}\cos(x^2) + \ln(-2x\sin(x^2))$

\boxed{(c)} $-2x\tan(x^2)$

(d) $2\ln(\cos(x))$

Question 15. *Shown is the graph of $f'(x)$, the <u>derivative</u> of $f(x)$:*



On what open interval(s) is $f(x)$ increasing?

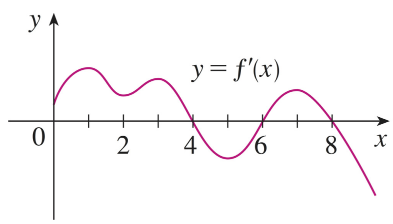
(a) $(0, 1) \cup (2, 3) \cup (5, 7)$

[(b)] $(0, 4) \cup (6, 8)$

(c) $(1.5, 2.5) \cup (4, 6)$

(d) $(0, \infty)$

Question 16. *Shown is the graph of $f'(x)$, the <u>derivative</u> of $f(x)$:*



On what open interval(s) is $f(x)$ concave up?

[(a)] $(0, 1) \cup (2, 3) \cup (5, 7)$

(b) $(0, 4) \cup (6, 8)$

(c) $(1.5, 2.5) \cup (4, 6)$

(d) $(0, \infty)$

Question 17. *At time t, the position of a particle moving along a straight line is given by the function*

$$s(t) = 2t^2 - 6t + 7.$$

Find the total distance traveled by the particle during the time interval $[0, 4]$.

(a) 7 *units*

(b) 8 *units*

(c) 15 *units*

[(d)] 17 *units*

Question 18. *The population of a bacteria culture doubles every half hour. If the initial population is 300, what is the population after t hours?*

(a) $P(t) = 300 \cdot 2^t$;

(b) $P(t) = 300 \cdot 2^{t/2}$;

[(c)] $P(t) = 300 \cdot 2^{2t}$;

(d) $P(t) = 300 + 2^t$.

Question 19. *Using implicit differentiation, find the slope of the tangent line to the curve*

$$x^3 + y^3 = 6xy$$

at the point $(3, 3)$.

[(a)] $m = -1$

(b) $m = 0$

(c) $m = 1$

(d) $m = 2$

Question 20. *Suppose that*

$$\lim_{x \to 5} f(x) = f(5).$$

Then which of the following statements must be true:

(a) f *has a jump discontinuity at* $x = 5$;

(b) f *has a removable discontinuity at* $x = 5$;

[(c)] f *is continuous at* $x = 5$;

(d) f *is differentiable at* $x = 5$.

Part II: Free Response (write answers & work in space provided)

Question 21. *Compute the following derivatives.*

(a) $\frac{d}{dx}\left[e^{5x}\tan^{-1}(3x+2)\right]$

(b) $\frac{d}{dx}\left[\frac{x}{(1-x^2)^{3/2}}\right]$ *(write answer as a fraction with simplified numerator)*

Solution. (a)

$$\frac{d}{dx}\left[e^{5x}\tan^{-1}(3x+2)\right] = 5e^{5x}\cdot\tan^{-1}(3x+2)+e^{5x}\cdot\frac{3}{1+(3x+2)^2}$$

(b)

$$\begin{aligned}\frac{d}{dx}\left[\frac{x}{(1-x^2)^{3/2}}\right] &= \frac{(1-x^2)^{3/2}\cdot 1 - x\cdot\frac{3}{2}(1-x^2)^{1/2}\cdot -2x}{(1-x^2)^3}\\ &= \frac{(1-x^2)^{1/2}[(1-x^2)+3x^2]}{(1-x^2)^3}\\ &= \frac{1+2x^2}{(1-x^2)^{5/2}}\end{aligned}$$

□

Question 22. *Find the equation of the tangent line to the ellipse*

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

at the point $(\sqrt{5}, 4/3)$.

Solution. Implicit differentiation:

$$\frac{2x}{9} + \frac{2yy'}{4} = 0 \implies \frac{2yy'}{4} = -\frac{2x}{9} \implies y' = -\frac{4x}{9y}$$

Slope of tangent line:

$$m = -\frac{4\sqrt{5}}{12} = -\frac{\sqrt{5}}{3}$$

Tangent line:

$$y - \frac{4}{3} = -\frac{\sqrt{5}}{3}(x - \sqrt{5}) \implies y = -\frac{\sqrt{5}}{3}x + 3$$

□

Question 23. *Find the largest and smallest values of the function*

$$f(x) = x - \sqrt{x}$$

on the interval $[0, 4]$.

Solution. Critical point:

$$f'(x) = 1 - \frac{1}{2\sqrt{x}} = 0 \implies 1 = \frac{1}{2\sqrt{x}} \implies \sqrt{x} = \frac{1}{2} \implies x = \frac{1}{4}$$

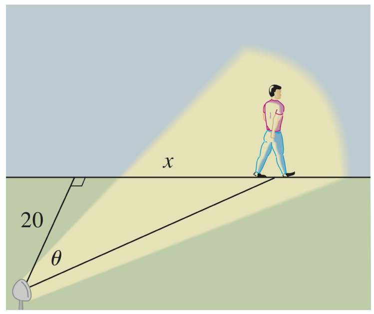
Comparison of y-values at endpoints and critical point:

$$f(0) = 0$$

$$f(1/4) = -1/4 \text{ (absolute minimum value)}$$
$$f(4) = 2 \text{ (absolute maximum value)}$$

□

Question 24. *A man walks along a straight path at a speed of* 3 *ft/s. A searchlight is located on the ground* 20 *ft from the path and is kept focused on the man.*



At what rate is the searchlight rotating when the man is 15 *ft from the point on the path closest to the searchlight?*

Solution. Given rate:

$$\frac{dx}{dt} = 3\ ft/s$$

Unknown rate:

$$\frac{d\theta}{dt} = \text{ ? when } x = 15\ ft$$

Equation between x and θ:

$$\tan(\theta) = \frac{x}{20}$$

Equation between dx/dt and $d\theta/dt$:

$$\sec^2(\theta) \cdot \frac{d\theta}{dt} = \frac{1}{20} \cdot \frac{dx}{dt}$$

Solution:

$$\tan(\theta) = \frac{15}{20} = \frac{3}{4} \implies \sec(\theta) = \frac{5}{4}$$

$$\left(\frac{5}{4}\right)^2 \cdot \frac{d\theta}{dt} = \frac{1}{20} \cdot 3 \implies \frac{d\theta}{dt} = 0.096\ rad/s$$

□

Question 25. *Use the first and second derivatives to sketch the graph the following curve:*

$$y = x^4 - 18x^2 + 81.$$

For full credit, your sketch must demonstrate

- *correct critical points and intervals of increase/decrease;*
- *correct concavity and inflection points;*
- *correct end behavior.*

Solution. Critical points:

$$y' = 4x^3 - 36x = 4x(x^2 - 9) = 0 \implies x = 0, \ \pm 3$$

$$(-3, 0), \ (0, 81), \ (3, 0)$$

Hyper-critical points:

$$y'' = 12x^2 - 36 = 12(x^2 - 3) = 0 \implies x = \pm\sqrt{3}$$

$$(-\sqrt{3}, 36), \ (\sqrt{3}, 36)$$

Sign chart:

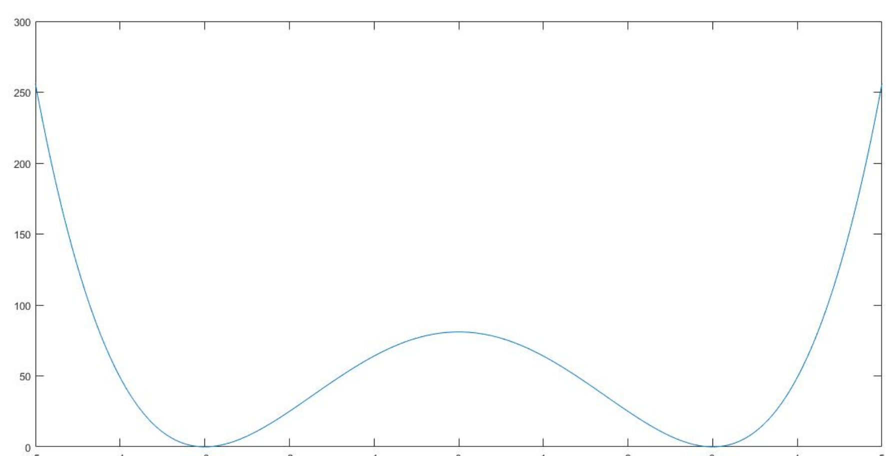
	$(-\infty, -3)$	$(-3, -\sqrt{3})$	$(-\sqrt{3}, 0)$	$(0, \sqrt{3})$	$(\sqrt{3}, 3)$	$(3, \infty)$
y'	$-$	$+$	$+$	$-$	$-$	$+$
y''	$+$	$+$	$-$	$-$	$+$	$+$
	decr/CU	incr/CU	incr/CD	decr/CD	decr/CU	incr/CU

End behavior:

$$\lim_{x\to\infty}(x^4 - 18x^2 + 81) = \lim_{x\to\infty} x^4\left(1 - \frac{18}{x^2} + \frac{81}{x^4}\right) = \infty$$

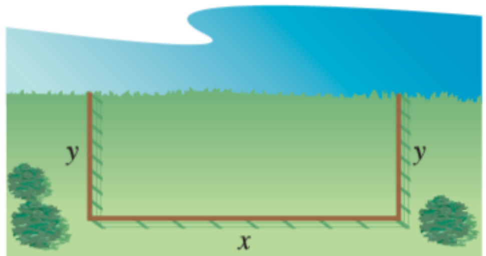
$$\lim_{x\to-\infty}(x^4 - 18x^2 + 81) = \lim_{x\to-\infty} x^4\left(1 - \frac{18}{x^2} + \frac{81}{x^4}\right) = \infty$$

Graph:



□

Question 26. *A farmer wishes to enclose* $245,000\ m^2$ *using a rectangular fence.*



What dimensions require the least amount of fencing, if no fencing is needed along the river?

Solution. Function to be optimized:

$$P = x + 2y$$

Eliminate x or y:

$$xy = 245000 \implies y = 245000/x$$
$$P = x + 490000/x$$

Optimize:

$$P' = 1 - 490000/x^2 = 0 \implies x^2 = 490000 \implies x = 700$$
$$P'' = 980000/x^3 \implies P''(700) = 1/350 > 0 \implies \text{min}$$
$$y = 245000/700 = 350$$

□

Question 27. *Find the degree 3 Taylor polynomial centered at $x = 1$ for the function*

$$f(x) = \ln(x),$$

and use it to approximate $\ln(1.4)$.

Solution.

$$T_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$$f(x) = \ln(x) \implies f'(x) = 1/x \implies f''(x) = -1/x^2 \implies f'''(x) = 2/x^3$$

$$f(1) = 0,\ f'(1) = 1,\ f''(1) = -1,\ f'''(1) = 2$$

$$T_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

$$\ln(1.4) = f(1.4) \approx T_3(1.4) = (0.4) - \frac{1}{2}(0.4)^2 + \frac{1}{3}(0.4)^3 = 0.341\overline{333}$$

□

Question 28. *Compute the following limit:*

$$\lim_{x\to 0^+} (1+2x)^{1/x}.$$

Solution.

$$L = \lim_{x\to 0^+} (1+2x)^{1/x} = 1^\infty$$

$$\begin{aligned} \ln(L) &= \lim_{x\to 0^+} \ln((1+2x)^{1/x}) = \lim_{x\to 0^+} \frac{\ln(1+2x)}{x} = \frac{0}{0} \\ &\overset{LH}{=} \lim_{x\to 0^+} \frac{\frac{2}{1+2x}}{1} = 2 \end{aligned}$$

$$\ln(L) = 2 \implies L = e^2$$

□

Question 29. *Find $f(x)$ if*

$$f''(x) = -2 + 12x - 12x^2, \ f'(0) = 12, \ f(0) = 4.$$

Solution.

$$f'(x) = -2x + 6x^2 - 4x^3 + C$$
$$12 = f'(0) = C$$
$$f'(x) = -2x + 6x^2 - 4x^3 + 12$$
$$f(x) = -x^2 + 2x^3 - x^4 + 12x + D$$
$$4 = f(0) = D$$
$$f(x) = -x^2 + 2x^3 - x^4 + 12x + 4$$

□

Question 30. *Prove the Product Rule of Differentiation. That is, prove that for differentiable functions $f(x)$ and $g(x)$,*

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

Solution.

$$\begin{aligned}
(f(x)g(x))' &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)[g(x+h) - g(x)] + [f(x+h) - f(x)]g(x)}{h} \\
&= \lim_{h\to 0} \left[f(x+h) \cdot \frac{g(x+h) - g(x)}{h} + \frac{f(x+h) - f(x)}{h} \cdot g(x) \right] \\
&= f(x)g'(x) + f'(x)g(x).
\end{aligned}$$

□