SM122 Calculus II Test I 2 Feb 2011

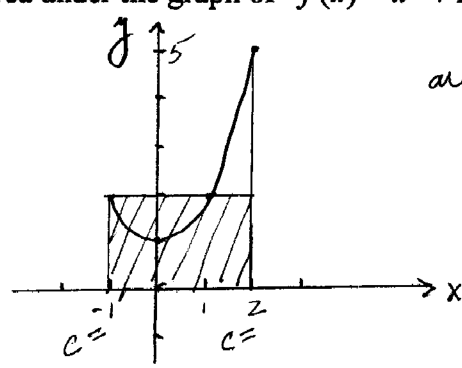
Name Solutions

1. (a) Find f_{ave}, the average value of the function $f(x) = x^2 + 1$ over the interval [-1 , 2].

$$f_{ave} = \frac{\int_a^b f(x)\,dx}{[b-a]} = \frac{\int_{-1}^{2} (x^2+1)\,dx}{[2-(-1)]} = \frac{\left(\frac{x^3}{3}+x\right)\Big|_{-1}^{2}}{3} = \frac{\left(\frac{8}{3}+2\right)-\left(\frac{-1}{3}-1\right)}{3}$$

$$= \frac{\frac{9}{3}+3}{3} = \frac{6}{3} = 2$$

(b) Sketch the graph of $f(x) = x^2 + 1$ over the interval [-1 , 2], and a rectangle whose area is the same as the area under the graph of $f(x) = x^2 + 1$.



area under curve = 6 = area of box.

(c) Find a number c where $f(c) = f_{ave}$ and show c *and* f_{ave} on your graph in (b).

$$f(c) = f_{ave} \Rightarrow c^2+1 = 2$$
$$\Rightarrow c^2 = 1$$
$$\Rightarrow c = -1 \text{ or } +1$$

(d) State carefully the Mean Value Theorem for integrals.

If f is continuous on a closed interval $[a,b]$, then there is at least one number c in $[a,b]$ where $f(c) = f_{ave} \left(= \frac{\int_a^b f(x)\,dx}{[b-a]}\right)$.

2. Evaluate the following by hand. Show every step.

(a) $\int \cos(e^{3x})\, e^{3x}\, dx$ let $u = e^{3x} \Rightarrow du = 3e^{3x}dx \Rightarrow \frac{1}{3}du = e^{3x}dx$

$= \frac{1}{3}\int \cos(u)\, du$

$= \frac{1}{3}\sin(u) + c$

$= \frac{1}{3}\sin(e^{3x}) + c$

(b) $\int [\tan(x)]^{2014} \sec^2(x)\, dx$ let $u = \tan(x) \Rightarrow du = \sec^2(x)\, dx$

$= \int u^{2014}\, du$

$= \dfrac{u^{2015}}{2015} + C$

$= \frac{1}{2015}\left[\tan(x)\right]^{2015} + c$

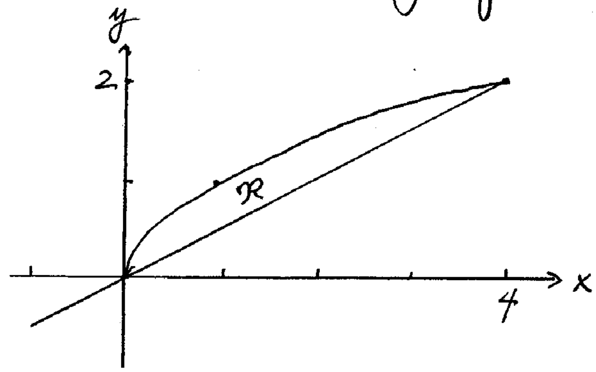
(c) $\int_0^2 \sqrt{9+x^4}\, x^3\, dx$ let $u = 9 + x^4 \Rightarrow du = 4x^3 dx \Rightarrow \frac{1}{4}du = x^3 dx$

$0 \xrightarrow{x} 2 \Rightarrow 9 \xrightarrow{u} 25$

$= \frac{1}{4}\int_9^{25} \sqrt{u}\, du$

$= \frac{1}{4}\cdot\frac{2}{3}u^{3/2}\Big|_9^{25} = \frac{1}{6}\left((\sqrt{25})^3 - (\sqrt{9})^3\right) = \frac{1}{6}(125 - 27) = \frac{98}{6} = \frac{49}{3}$

3. The region $\Re$ on the right is bounded by the curves $y=\sqrt{x}$ and $y=\frac{1}{2}x$.

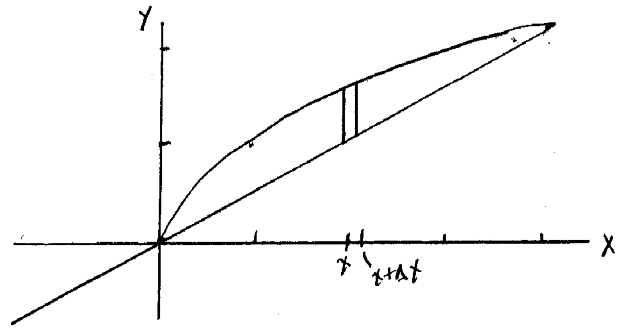


Set up **BUT DO NOT EVALUATE** integrals necessary to find the following:
(Draw figures and show the element of area or volume you are using to set up the integral.)

(a) The area of the region $\Re$.

$\Delta A =$ area of 1 rect $= (\sqrt{x} - x/2)\,\Delta x$

$$A = \int_0^4 (\sqrt{x} - x/2)\,dx$$



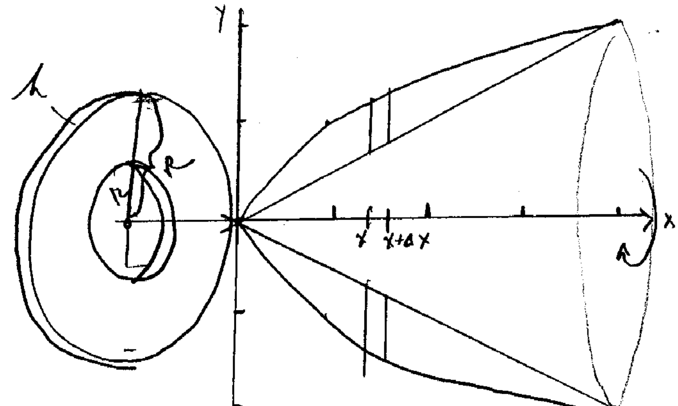
(b) The volume of the solid obtained by rotating the region $\Re$ about the x axis.

$\Delta V =$ vol of 1 disk

$= \pi R^2 h - \pi r^2 h$

$= \pi (R^2 - r^2) h = \pi\left((\sqrt{x})^2 - \left(\frac{x}{2}\right)^2\right)\Delta x$

$$V = \pi \int_0^4 \left[x - \frac{x^2}{4}\right] dx$$



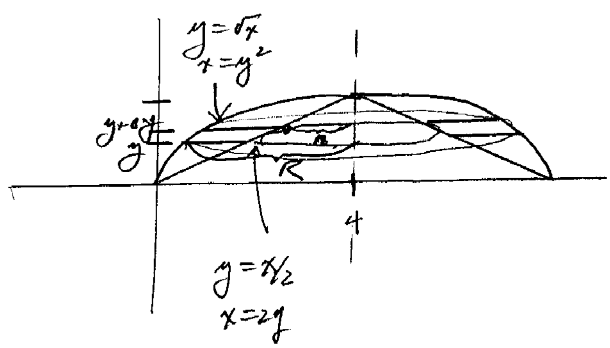
(c) The volume of the solid obtained by rotating the region $\Re$ about the vertical line $x=4$.

$\Delta V =$ vol of 1 disk

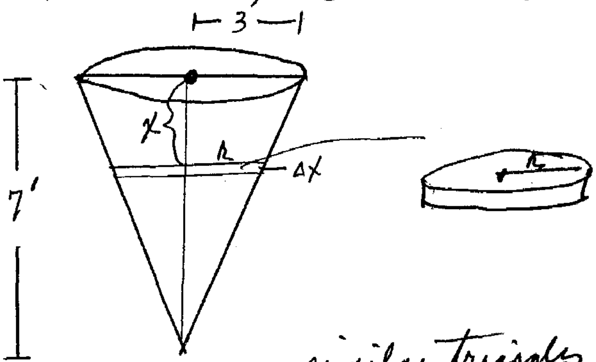
$= \pi (R^2 - r^2) h$

$= \pi \left((4-y^2)^2 - (4-2y)^2 \right) \Delta y$

$$V = \pi \int_0^2 \left[(4-y^2)^2 - (4-2y)^2\right] dy$$



4. A tank full of water has the shape of a right circular cone of height 7 ft and radius of 3 ft. Set up **BUT DO NOT EVALUATE** an integral to find the work necessary to pump the water out of the top of the tank. (Water has a weight density of 62.5 lbs/ft^3). (Draw and label the segment of water you are using to set up your integral.)



ΔW = work to lift 1 disk of water
= force · distance
= weight · x ft
$= \left(62.5 \frac{\text{lbs}}{\text{ft}^3}\right)(\text{vol ft}^3)(x \text{ ft})$
$= 62.5\,(\pi r^2 \cdot \Delta x)\, x$ ft-lbs
$= 62.5\left(\pi\left(\frac{3}{7}(7-x)\right)^2 x\, \Delta x\right)$ ft-lbs.

similar triangles

$\frac{r}{7-x} = \frac{3}{7}$

$\Rightarrow r = \frac{3}{7}(7-x)$

total $W = 62.5\pi \frac{9}{49} \int_0^7 (7-x)^2 x\, dx$ ft-lbs.