Calculus II Test II March 4, 2011

Name SOLUTIONS Prof. D'Archangelo

1. [30 pts] Evaluate the following integrals by hand. Show all steps.

(a) $\int x\cos(5x)\,dx$

let $u = x \Rightarrow du = 1dx$

let $dv = \cos(5x)dx \Rightarrow v = \frac{\sin(5x)}{5}$

$$\int u\,dv = uv - \int v\,du$$

$$= x\frac{\sin(5x)}{5} - \int \frac{\sin(5x)}{5}dx$$

$$= \frac{1}{5}x\sin(5x) + \frac{1}{25}\cos(5x) + C$$

(b) $\int x\ln(x)\,dx$

let $u = \ln(x) \Rightarrow du = \frac{1}{x}dx$

let $dv = x\,dx \Rightarrow v = \frac{x^2}{2}$

$$= \int \ln(x)\,x\,dx = \int u\,dv = uv - \int v\,du$$

$$= \ln(x)\cdot\left(\frac{x^2}{2}\right) - \int \frac{x^{\cancel{2}}}{2}\cdot\frac{1}{\cancel{x}}dx$$

$$= \frac{x^2}{2}\ln(x) - \frac{x^2}{4} + C$$

(c) $\int \frac{3x^2+2}{(x)(x^2+1)}dx$

$$\frac{3x^2+2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$\Rightarrow 3x^2+2 = A(x^2+1) + (Bx+C)x$

let $x=0 \Rightarrow$ $2 = A(1)+0 \Rightarrow \boxed{A=2}$

so $3x^2+12 = 2(x^2+1)+(Bx+C)x$

$3x^2+0x+12 = (2+B)x^2 + Cx + 2$

$\Rightarrow 2+B=3 \Rightarrow \boxed{B=1}$

$\boxed{C=0}$

$$= \int \frac{2}{x}dx + \int \frac{x}{x^2+1}dx$$

$$= 2\ln|x| + \frac{1}{2}\int \frac{\overbrace{2x\,dx}^{du}}{\underbrace{x^2+1}_{u}}$$

$$= 2\ln|x| + \frac{1}{2}\int \frac{1}{u}du$$

$$= 2\ln|x| + \frac{1}{2}\ln|x^2+1| + C$$

2. [10 pts] Prove the integration by parts formula $\int u\,dv = uv - \int v\,du$ by first differentiating $u(x)v(x)$ and then integrating the resulting formula.

① $[u(x)\cdot v(x)]' = u'(x)v(x) + u(x)v'(x)$ (product rule)

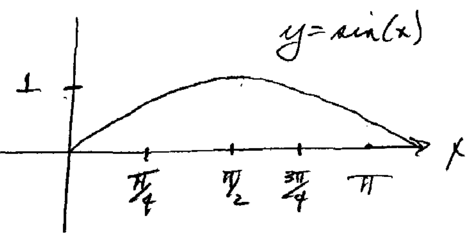
② $\int [u(x)\cdot v(x)]'\,dx = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx$ (integrate both sides)

③ $u(x)v(x) = \int v(x)\frac{du}{dx}\,dx + \int u(x)\frac{dv}{dx}\,dx$ $\quad \int f'(x)\,dx = f(x)$ + notation

④ $uv = \int v\,du + \int u\,dv$ (notation)

⑤ $\int u\,dv = uv - \int v\,du$ (subtraction)

3. [10 pts] Use S_4 (Simpson's rule with 4 subdivisions) to approximate the area under one hump of the function $\sin(x)$ over the interval $[0, \pi]$. Give your answer to 3 decimal places. How much error is in your approximation?

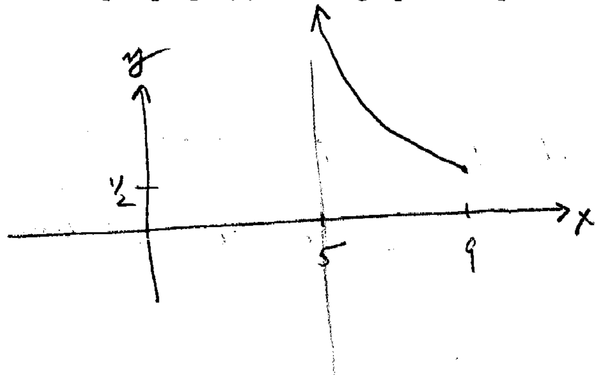


$$S_4 = \frac{\Delta x}{3}\left[f(0) + 4f\left(\tfrac{\pi}{4}\right) + 2f\left(\tfrac{\pi}{2}\right) + 4f\left(\tfrac{3\pi}{4}\right) + f(\pi)\right]$$

$$= \frac{\pi/4}{3}\left[0 + 4\left(\frac{\sqrt{2}}{2}\right) + 2(1) + 4\left(\frac{\sqrt{2}}{2}\right) + 0\right]$$

$$= \frac{\pi}{12}\left[4\sqrt{2} + 2\right] \doteq 2.004559 \doteq 2.005$$

4. [10 pts] (a) Use a graph to explain why $\int_5^9 \frac{1}{\sqrt{x-5}}\,dx$ is an improper integral.



The function is not continuous on [5, 9]. It is undefined at $x=5$ and has a vertical asymptote there.

(b) Evaluate the integral by first writing it as a limit of a proper integral. Show all steps.

$$\int_5^9 \frac{1}{\sqrt{x-5}}\,dx = \lim_{t\to 5^+}\left[\int_t^9 (x-5)^{-1/2}\,dx\right]$$

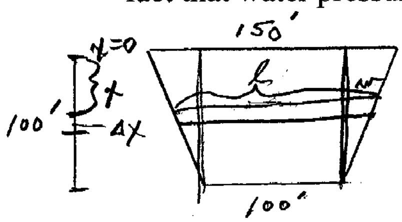
$$= \lim_{t\to 5^+}\left[2(x-5)^{1/2}\Big|_t^9\right]$$

$$= \lim_{t\to 5^+}\left[2\sqrt{9-5} - 2\sqrt{t-5}\right]$$

$$4 \quad - \quad 0$$

$$= 4$$

5. [10 pts] A dam with water to its top is in the shape of the trapezoid shown below. **Set up, but do not evaluate** an integral to find the hydrostatic force pushing against the dam. Use the fact that water pressure at a depth of x feet is $62.5x$ lbs/ft^2.



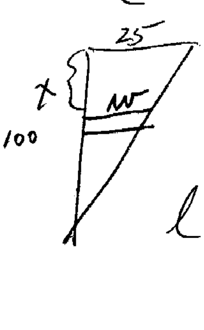
ΔHF = hydrostatic force against 1 long strip

= (pressure)(area)

$= (62.5x)\frac{\text{lbs}}{\text{ft}^2}(l\,\Delta x)\text{ft}^2$

$= (62.5x)\left(150 - \frac{x}{2}\right)$ lbs.

$l = 100 + 2w$



$$\frac{w}{100-x} = \frac{25}{100}$$

$$w = \frac{1}{4}(100-x)$$

$$l = 100 + \frac{2}{4}(100-x)$$

$$= 100 + 50 - x/2$$

$$= 150 - \frac{x}{2}$$

$$\text{total } HF = \int_0^{100} (62.5x)\left(150 - \frac{x}{2}\right)dx \text{ lbs}$$

6. [10 pts] If $y(t)$ is the temperature of an object in a room of constant temperature R, then the differential equation for Newton's Law of Cooling is $\frac{dy}{dt} = k(y - R)$.

a) Explain what this differential equation means in plain English.

At any time t, the rate of change of the temperature of the object with respect to time is proportional to the difference between the object's temp and the room temp.

b) A cup of coffee has temperature $150^\circ F$ when taken from a microwave and placed in a room of $70^\circ F$. Ten minutes later the coffee is $110^\circ F$. Solve a differential equation to find $y(t)$, the temperature of the coffee at time t. When will the temperature of the coffee be $90^\circ F$?

$y(t)$ = temp of coffee

$y(0) = 150^\circ F$

$y(10) = 110^\circ F$

$$\frac{dy}{dt} = k(y-70)$$

separate variables $\left(\frac{1}{y-70}\right)dy = k\,dt$

$$\int \frac{1}{y-70}\,dy = \int k\,dt$$

$$e^{(\ln|y-70|)} = e^{(kt+C)}$$

$$|y-70| = e^{C} \cdot e^{kt}$$

$$y-70 = (\pm e^{C})e^{kt} = Ae^{kt}$$

$$\boxed{y = 70 + Ae^{kt}}$$

$y(0) = 150$ $\Rightarrow$ $150 = 70 + Ae^{0} \Rightarrow A = 80$

so $\boxed{y = 70 + 80e^{kt}}$

$y(10) = 110 \Rightarrow \cancel{110}\ 40 = \cancel{70} + 80e^{k10}$

$\Rightarrow \frac{1}{2} = e^{10k} \Rightarrow \ln\left(\frac{1}{2}\right) = 10k \doteq -.069$

so $\boxed{y = 70 + 80e^{-.069t}}$

when $y = 90$ find t

$\cancel{90}\ 20 = \cancel{70} + 80e^{-.069t}$

$\frac{1}{4} = e^{-.069t} \Rightarrow \frac{\ln(1/4)}{-.069} = \cancel{-.069}t \doteq 20$ mins

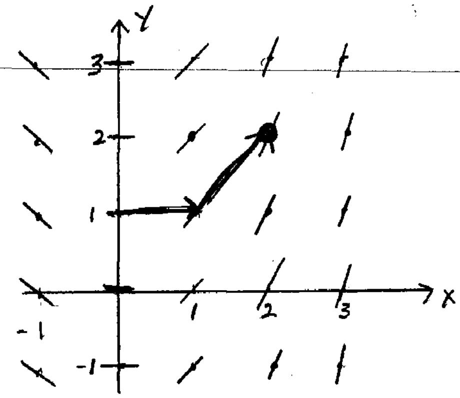
Calculus II Test II take home questions March 4, 2011 P1 of 2

Name SOLUTIONS Prof. D'Archangelo

I have not used assistance from any other person.

1. A portion of a direction field for a differential equation $y' = f(x, y)$ is drawn below. If $y(0) = 1$, use two steps of Euler's method to approximate $y(2)$.

Hint: You can solve the problem with a ruler.

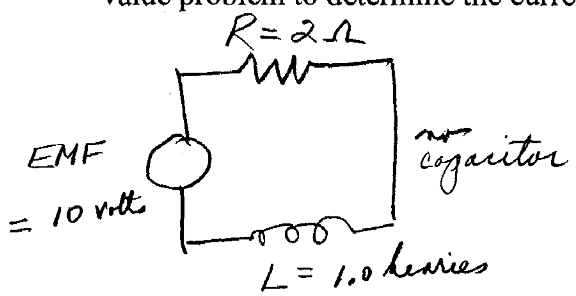


graphically $y(0)=1$ (slope $=0$)

$\Rightarrow y(1)=1$ (slope $=1$)

$\Rightarrow \boxed{y(2)=2}$

2. A simple series circuit consists of a 1.0 henry inductor, a 2.0 ohm resistor, and a constant EMF $E(t) = 10$ volts. If the initial current in the circuit is 20 amperes, set up and solve an initial value problem to determine the current $I(t)$ for $t > 0$.



$$LI' + RI + \cancel{\tfrac{1}{C}Q} = EMF$$

$$1I' + 2I = 10$$

$$\frac{dI}{dt} = 10 - 2I$$

separate variables $\quad \frac{1}{10-2I}\,dI = 1\,dt$

$$\int \frac{1}{10-2I}\,dI = \int 1\,dt$$

$$\Rightarrow \frac{\ln|10-2I|}{-2} = (t+c)(-2)$$

$$e \qquad e$$

$$|10-2I| = e^{-2t}\cdot e^{A} \Rightarrow \quad 10-2I = Ce^{-2t}$$

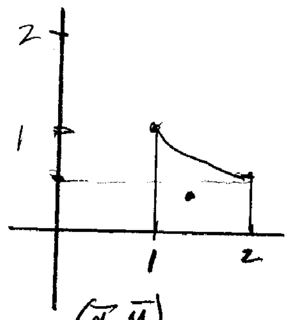
OVER

$$\Rightarrow \quad 2I = 10 - Ce^{-2t} \Rightarrow I = 5 - Ke^{-2t}$$

$$I(0) = 20 \Rightarrow 20 = 5 - Ke^{0} \Rightarrow K = -15 \Rightarrow I = 5 + 15e^{-2t}$$

P2 of 2.

3. Sketch the region bounded by the curves $y = 1/x$, $x = 1$, and $x = 2$. Find the exact centroid of the region and plot it on your graph.



$(\bar{x}, \bar{y})$

$= (1.44, .36)$

$$\bar{x} = \frac{\int_1^2 x\left(\frac{1}{x}\right)dx}{\int_1^2 \frac{1}{x}\,dx} = \frac{\int_1^2 1\,dx}{\ln|x|\,\big|_1^2} = \frac{x\,\big|_1^2}{\ln|2| - \ln|1|} = \frac{1}{\ln(2)} \doteq 1.44$$

$$\bar{y} = \frac{\int_0^1 \frac{1}{2}\left(\frac{1}{x}\right)^2 dx}{\ln(2)} = \frac{\frac{1}{2}\int_1^2 x^{-2}\,dx}{\ln(2)} = \frac{-\frac{1}{2}x^{-1}\,\big|_1^2}{\ln(2)} = \frac{-\frac{1}{2}\left(\frac{1}{2} - \frac{1}{1}\right)}{\ln(2)}$$

$$= \frac{\frac{1}{4}}{\ln(2)} = .36$$