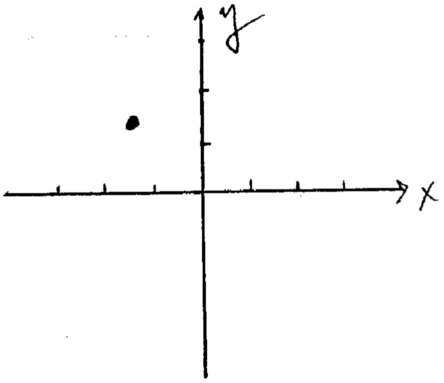
Calculus II Test III April 5, 2011

Name SOLUTIONS Prof. D'Archangelo

1. (a) Sketch the point whose polar coordinates are $(r,\theta) = (2, 3\pi/4)$ and find the point's Cartesian coordinates (x , y).

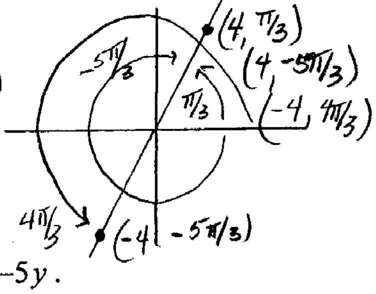


$x = r\cos(\theta) = 2\cos(3\pi/4) = 2\left(-\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$

$y = r\sin(\theta) = 2\sin\left(\frac{3\pi}{4}\right) = 2\frac{\sqrt{2}}{2} = \sqrt{2}$

(b) Four sets of polar coorinates (r,θ) are given. Three of them represent the <u>same point</u>. Which set of polar coordinates represents a <u>different point</u> from the other three? (Circle your answer.)

(i) $(4, \pi/3)$ (ii) $(4, -5\pi/3)$ (iii) $(-4, 4\pi/3)$ [✓ (iv) $(-4, -5\pi/3)$]



(c) Find a polar equation for the curve represented by the Cartesian equation $x^2 + y^2 = -5y$.

$x = r\cos(\theta)$

$y = r\sin(\theta)$

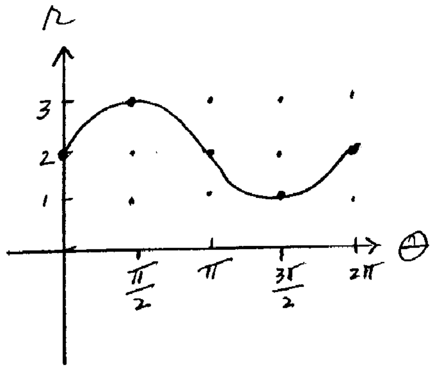
$r^2\cos^2(\theta) + r^2\sin^2(\theta) = -5r\sin(\theta)$

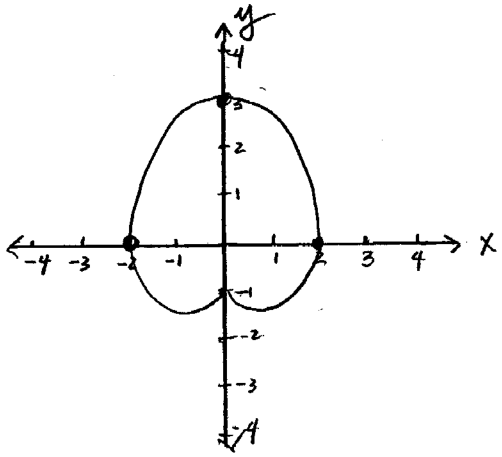
$r^2\underbrace{(\cos^2(\theta) + \sin^2(\theta))}_{1} = -5r\sin(\theta)$

$r^2 = -5r\sin(\theta)$

$r = -5\sin(\theta)$

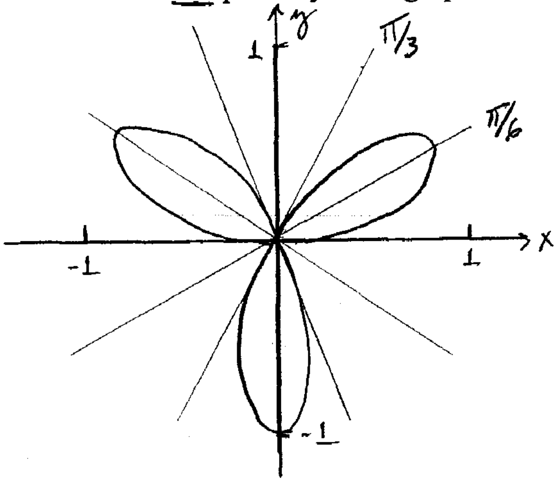
2. The curve below shows the graph of r as a function of θ in the $\theta - r$ plane. Use it to sketch the corresponding polar curve in the $x - y$ plane.





3. (a) Sketch the 3-petal curve $r = \sin(3\theta)$ in the x – y plane (you may use your calculator) and
(b) find the area in <u>one</u> petal by setting up and evaluating an integral (you may use your calculator).

(a)



(b) AREA

$$= \cancel{2}\int_0^{\pi/6} \cancel{\tfrac{1}{2}} r^2 \, d\theta$$

$$= \int_0^{\pi/6} [\sin(3\theta)]^2 \, d\theta$$

$$= \int_0^{\pi/6} \frac{1 - \cos(6\theta)}{2} \, d\theta$$

$$= \left[\tfrac{1}{2}\theta - \frac{\sin(6\theta)}{12}\right]_0^{\pi/6} = \pi/12$$

4. Find the sum of the geometric series $2 + 4/3 + 8/9 + \ldots$.

$$a + ar + ar^2 + \cdots \quad \text{where } a = 2,\ r = 2/3;\ |r| < 1$$

$$= \frac{a}{1-r} = \frac{2}{1 - 2/3} = \frac{2}{1/3} = 6$$

5. Write out the first three terms of each of the following series and tell whether each of the series converges or diverges (you must justify your answers.)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{-4n}{5n+1} = \frac{-4}{6} - \frac{8}{11} - \frac{12}{16} - \cdots$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{(4n)/n}{(5n+1)/n} = \lim_{n\to\infty} \frac{4}{5 + \underset{0}{1/n}} = \frac{4}{5} \neq 0$$

$\Rightarrow$ series diverges by the <u>divergence test</u>

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{3n-1} = \frac{-1}{2} + \frac{1}{5} - \frac{1}{8} + \cdots$

alt series test

① signs alternate ✓

② $|a_{n+1}| = \dfrac{1}{3(n+1)-1} < \dfrac{1}{3n-1} = |a_n|$ ✓

③ $\displaystyle\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{3n-1} = 0$ ✓

$\Rightarrow$ series converges.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ ("harmonic series")

$$= 1 + \frac{1}{2} + \underbrace{\overset{>\frac{1}{4}}{\frac{1}{3}} + \frac{1}{4}}_{>\frac{1}{2}} + \underbrace{\overset{>\frac{1}{8}}{\frac{1}{5}} + \overset{>\frac{1}{8}}{\frac{1}{6}} + \overset{>\frac{1}{8}}{\frac{1}{7}} + \frac{1}{8}}_{>1/2} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}}_{>1/2} + \cdots$$

$s_1 = 1$

$s_2 = 1 + \frac{1}{2}$

$s_4 > 2$

$s_8 > 2\frac{1}{2}$

$s_{16} > 3$

definition: The sequence of partial sums diverges to ∞ $\Rightarrow$ Series diverges

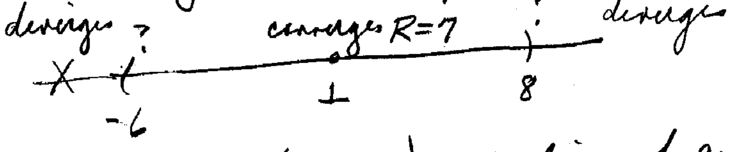
6. Find the open interval of convergence (no endpts) and radius of convergence for the series $\sum_{n=1}^{\infty} \frac{n(x-1)^n}{7^n}$.

ratio test

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{(n+1)\overset{(x-1)}{\cancel{(x-1)^{n+1}}}}{\underset{7}{\cancel{7^{n+1}}}} \right| \left| \frac{\cancel{7^n}}{n\cancel{(x-1)^n}} \right|$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right) \frac{|x-1|}{7} = \lim_{n\to\infty} (1+1/n) \frac{|x-1|}{7} = 1 \frac{|x-1|}{7}$$

if $L<1$ the series converges $\Leftrightarrow \frac{|x-1|}{7} < 1 \Rightarrow |x-1| < 7$



$\Rightarrow$ open interval of convergence $(-6, 8)$; radius of convergence $R = 7$.

7. Show that the Maclaurin series for $f(x) = \sin(x)$ is $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ by finding the Maclaurin coefficients directly.

$f(x) = \sin(x) \Rightarrow f(0) = 0$

$f'(x) = \cos(x) \Rightarrow f'(0) = 1$

$f''(x) = -\sin(x) \Rightarrow f''(0) = 0$

$f'''(x) = -\cos(x) \Rightarrow f'''(0) = -1$

$f^{IV}(x) = \sin(x) \Rightarrow f^{IV}(0) = 0$

$\vdots$

repeats

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{IV}(0)\frac{x^4}{4!} + \ldots$$

$$\sin(x) = 0 + 1x + 0\frac{x^2}{2!} - 1\frac{x^3}{3!} + 0\frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

$$\text{so } \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

8. (a) Use the Maclaurin series in problem 7 to find the Maclaurin series for $f(x) = x^2 \sin(x^2)$. Show the first four non-zero terms of your answer. Extra credit for writing the answer in summation notation.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin(x^2) = (x^2) - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$$

$$x^2 \sin(x^2) = x^4 - \frac{x^8}{3!} + \frac{x^{12}}{5!} - \frac{x^{16}}{7!} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n}}{(2n-1)!} \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+4}}{(2n+1)!}$$

(b) Find $\int_0^{0.5} x^2 \sin(x^2)\,dx$ to within .001 using your series from part (a).

$$= \int_0^{0.5} \left[x^4 - \frac{x^8}{3!} + \frac{x^{12}}{5!} - \frac{x^{16}}{7!} + \cdots \right] dx$$

$$= \frac{x^5}{5} - \frac{x^9}{3!9} + \frac{x^{13}}{5!\cdot 13} - \frac{x^{17}}{7!\cdot 17} \Big|_0^{0.5}$$

$$= \frac{(0.5)^5}{5} - \frac{(0.5)^9}{54} + \cdots$$

$$= \boxed{.00625} / - .000036$$

9. Let $f(x) = \cos(x)$. Find the first four non-zero terms of the Taylor series for $f(x)$ centered at $a = \pi/4$.

$f(x) = \cos(x) \Rightarrow f(\pi/4) = \sqrt{2}/2$

$f'(x) = -\sin(x) \Rightarrow f'(\pi/4) = -\sqrt{2}/2$

$f''(x) = -\cos(x) \Rightarrow f''(\pi/4) = -\sqrt{2}/2$

$f'''(x) = \sin(x) \Rightarrow f'''(\pi/4) = \sqrt{2}/2$

$f^{IV}(x) = \cos(x)$

repeats

$$\cos(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots$$

$$\cos(x) = f(\tfrac{\pi}{4}) + f'(\tfrac{\pi}{4})(x - \tfrac{\pi}{4}) + \frac{f''(\frac{\pi}{4})(x-\frac{\pi}{4})^2}{2!} + \frac{f'''(\pi/4)(x-\frac{\pi}{4}}{3!}$$

$$\cos(x) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(x - \pi/4) - \frac{\sqrt{2}}{2}\frac{(x-\pi/4)^2}{2!} + \frac{\sqrt{2}}{2\cdot 3!}(x-\pi/4)^3 + \cdots$$