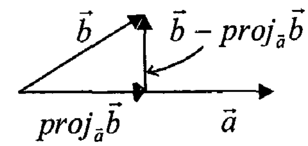
Calculus II Test IV April 27, 2011

Name ~~Solutions~~ Prof. D'Archangelo

(10) 1. Prove that for any two non-zero vectors $\vec{a}$ and $\vec{b}$, $\vec{b} - proj_{\vec{a}}\vec{b}$ is perpendicular to $\vec{a}$.
Show all steps and give reasons.



$\vec{a} \cdot (\vec{b} - proj_{\vec{a}}\vec{b})$ (want to show $=0$)

$= \vec{a} \cdot \left[\vec{b} - \left(\vec{b} \cdot \frac{\vec{a}}{|\vec{a}|}\right)\frac{\vec{a}}{|\vec{a}|}\right]$ (defn of $proj_{\vec{a}}\vec{b}$)

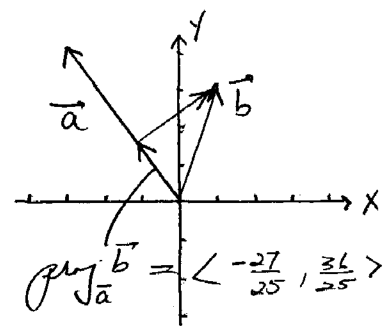
$= \vec{a} \cdot \vec{b} - \frac{(\vec{b} \cdot \vec{a})}{|\vec{a}|^2}(\vec{a} \cdot \vec{a})$ (distribution of dot prod)

$= \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a}$ $\quad (\vec{a} \cdot \vec{a} = |\vec{a}|^2)$

$= 0$ (dot prod is commutative)

(20) 2. Consider the two vectors $\vec{a} = \langle -3,4 \rangle$ and $\vec{b} = \langle 1,3 \rangle$.

a) Sketch both vectors on the axes to the right.
(with tails at the origin)



(b) Find the angle (degrees) between $\vec{a}$ and $\vec{b}$.

$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos(\theta)$

$\langle -3,4 \rangle \cdot \langle 1,3 \rangle = \sqrt{9+16}\,\sqrt{1+9}\,\cos(\theta)$

$9 = 5\sqrt{10}\cos(\theta) \Rightarrow \theta = \cos^{-1}\left(\frac{9}{5\sqrt{10}}\right) = 55.3^\circ$

(c) Find the scalar component of $\vec{b}$ onto $\vec{a}$, ($comp_{\vec{a}}\vec{b}$).

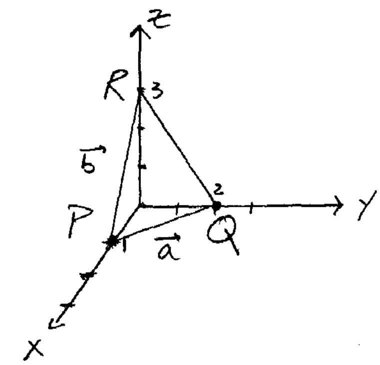
$\vec{b} \cdot \hat{a} = \langle 1,3 \rangle \cdot \frac{\langle -3,4 \rangle}{5} = \frac{-3+12}{5} = \frac{9}{5}$

(d) Find the vector projection of $\vec{b}$ onto $\vec{a}$, ($proj_{\vec{a}}\vec{b}$) <u>and</u> sketch it in your graph above.

$proj_{\vec{a}}\vec{b} = (\vec{b} \cdot \hat{a})\hat{a} = \frac{9}{5}\frac{\langle -3,4 \rangle}{5} = \langle -\frac{27}{25}, \frac{36}{25} \rangle$

(15) 3. Consider the three points in space: P(1, 0, 0), Q(0, 2, 0), and R(0, 0, 3).

a) Sketch the triangle in 3-d space on the axes to the right.



b) Find the area of the triangle ΔPQR.

$\vec{a} = \overrightarrow{PQ} = \langle -1, 2, 0 \rangle$

$\vec{b} = \overrightarrow{PR} = \langle -1, 0, 3 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = \langle 6, 3, 2 \rangle$$

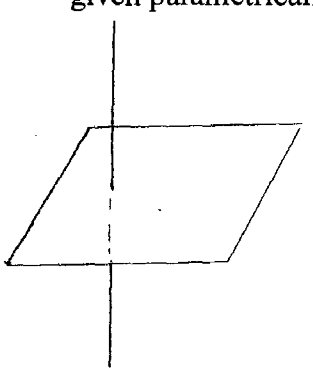
area of $\Delta = \frac{1}{2} |\vec{a} \times \vec{b}| = \frac{1}{2} \sqrt{36+9+4} = 7/2$

c) Find parametric equations for the line going through the points P and R.

The line contains $P(1,0,0)$ and is parallel to $\overrightarrow{PR} = \langle -1, 0, 3 \rangle$

$$\Rightarrow \begin{aligned} x &= 1 + (-1)t \\ y &= 0 + 0(t) \\ z &= 0 + 3(t) \end{aligned} \Rightarrow \begin{aligned} x &= 1 - t \\ y &= 0 \\ z &= 3t \end{aligned} ; \quad -\infty < t < \infty .$$

(10) 4. Find an equation for the plane going through the origin and perpendicular to the line given parametrically by x = 1 + 2t, y = 1 + 3t, z = -2 + 4t.



$\vec{N} = \langle 2, 3, 4 \rangle$ is normal to the plane

$P(0,0,0)$ is a pt on the plane

$\Rightarrow 2(x-0) + 3(y-0) + 4(z-0) = 0$

$\Rightarrow 2x + 3y + 4z = 0$

(15) 5. State whether each of the following is a vector, a scalar, or makes no sense:

(a) $(\vec{a} \cdot \vec{b})\vec{c}$ = (scalar) vector = vector

(b) $(\vec{a} \times \vec{b}) \cdot |\vec{c}|$ = vect • scal = no sense

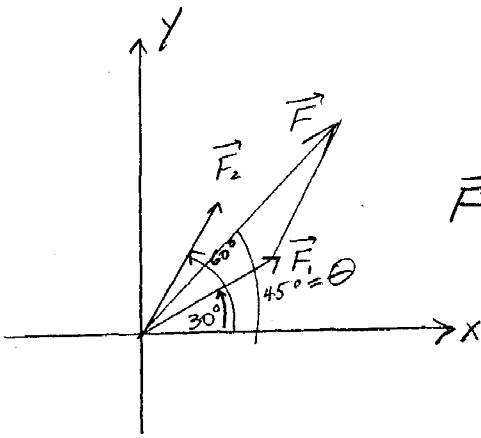
(c) $(\vec{a} \times \vec{b}) \times \vec{c}$ ⇒ vector x vect = vect

(d) $\vec{a} \cdot (\vec{b} \times \vec{c})$ = vect • vector = scalar

(e) $(\vec{a} \cdot \vec{b}) \times \vec{c}$ = scal x vect = no sense

(f) $(\vec{a} \cdot \vec{b}) + |\vec{c}|$ = scalar + scalar = scalar

(10) 6. Two forces $\vec{F}_1$ and $\vec{F}_2$ each with magnitude 10 Newtons make angles of 30° and 60° respectively with the positive x –axis in the 2-d plane. If $\vec{F} = \vec{F}_1 + \vec{F}_2$, find the magnitude and the angle $\vec{F}$ makes with the positive x – axis.



$\vec{F}_1 = \langle 10\cos(30^\circ), 10\sin(30^\circ) \rangle = \langle 10\frac{\sqrt{3}}{2}, 10(\frac{1}{2}) \rangle = \langle 5\sqrt{3}, 5 \rangle$

$\vec{F}_2 = \langle 10\cos(60^\circ), 10\sin(60^\circ) \rangle = \langle 5, 5\sqrt{3} \rangle$

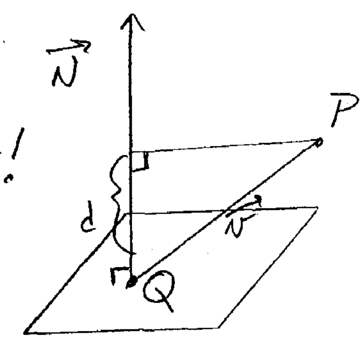
$\vec{F} = \vec{F}_1 + \vec{F}_2 = \langle 5\sqrt{3}+5, 5+5\sqrt{3} \rangle = (5\sqrt{3}+5)\langle 1, 1 \rangle$

$|\vec{F}| = (5\sqrt{3}+5)\sqrt{2} = 19.3$

$\theta = \tan^{-1}\left(\frac{5\sqrt{3}+5}{5\sqrt{3}+5}\right) = \tan^{-1}(1) = 45^\circ$

(10) 7. Find the distance between the point P(1, 1, 1) and the plane x + 2y + 3z = 1.

Hint!



$Q(1,0,0)$ is in the plane ; $\vec{N} = \langle 1,2,3 \rangle$ is $\perp$ the plane.

$\vec{v} = \vec{QP} = \langle 0,1,1 \rangle$

$d = |\vec{v} \cdot \hat{N}| = \left| \langle 0,1,1 \rangle \cdot \frac{\langle 1,2,3 \rangle}{\sqrt{1+4+9}} \right| = \frac{2+3}{\sqrt{14}} = \frac{5}{\sqrt{14}}$

$= \frac{5\sqrt{14}}{14} \doteq 1.34$

(10) 8. Where does the line given parametrically by x = 3 + t, y = 2 + 3t, z = -2 + 2t intersect the plane x + y – z = 3?

$x + y - z = 3$

$\Rightarrow (3+t) + (2+3t) - (-2+2t) = 3$

$\Rightarrow 7 + 2t = 3$

$\Rightarrow 2t = -4$

$\Rightarrow t = -2 \quad \Longrightarrow \quad x = 3 + (-2) = 1$

$y = 2 + (3)(-2) = -4 \Rightarrow P(1, -4, -6)$

$z = -2 + 2(-2) = -6$