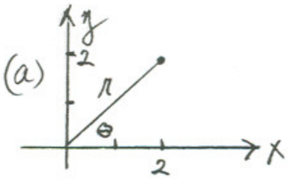
Calculus II Chapters 10 + 11 Practice Problem Solutions

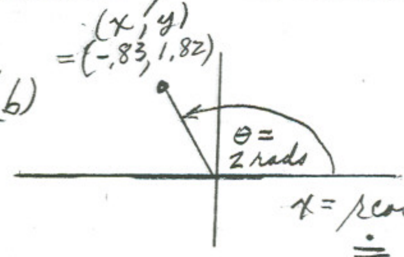
1. (a)



$x=2$, $y=2$ $\Big\}$ $r=\sqrt{x^2+y^2} = \sqrt{8} = 2\sqrt{2}$

$\theta = \tan^{-1}(y/x) = \tan^{-1}(1) = \pi/4$

(b)

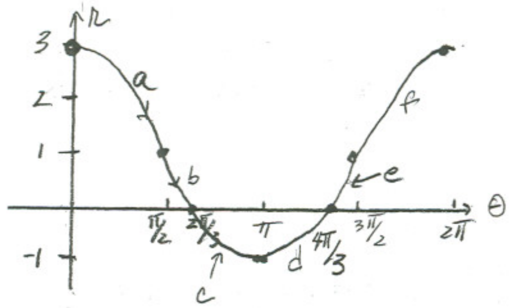


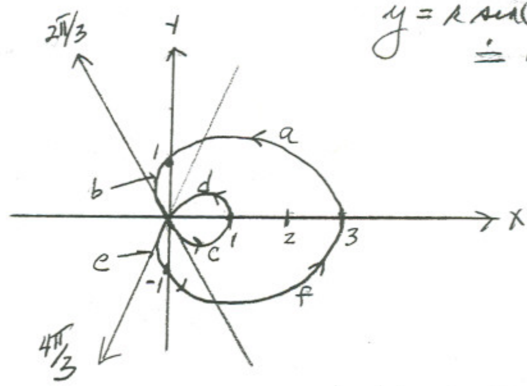
$r=2$
$\theta = 2$ rads

$x = r\cos(\theta) = 2\cos(2) \doteq -.832$

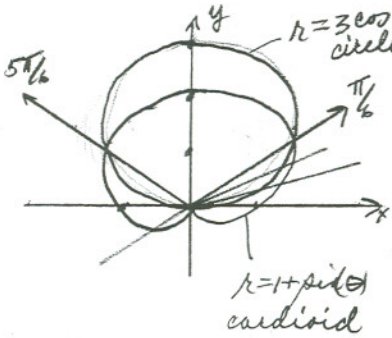
$y = r\sin(\theta) = 2\sin(2) \doteq 1.819$

2.





3. (a)



$3\sin(\theta) = 1+\sin(\theta)$

$2\sin(\theta) = 1$

$\sin(\theta) = 1/2 \Rightarrow \theta = \pi/6,\ 5\pi/6$

(b) inside $r = 1+\sin(\theta)$
outside $r = 3\sin(\theta)$

$$= 2\int_{-\pi/2}^{\pi/6} \frac{1}{2}\left(1+\sin(\theta)\right)^2 d\theta - 2\int_0^{\pi/6} \frac{1}{2}\left(3\sin(\theta)\right)^2 d\theta = \pi/4 \text{ (calculator)}$$

4. (a) $\displaystyle \lim_{n\to\infty} \frac{(n+1)/n}{(5n-1)/n} = \lim_{n\to\infty} \frac{1+1/n}{5-1/n} = \frac{1}{5}$

(b) $\displaystyle \sum_{n=1}^{\infty} \frac{n+1}{5n-1}$ diverges
by divergence test: $\lim_{n\to\infty} a_n \neq 0$

5. distance $= 10 + \underbrace{10+5+2.5+1.25+\cdots}_{\text{geometric series with } a=10,\ r=1/2} = 10 + \frac{a}{1-r} = 10 + \frac{10}{1-1/2} = 10+20 = 30'$

10 ↓ 5 ↑ ↓ 5 2.5 2.5 ↑↓

6. $\displaystyle \sum_{n=1}^{\infty} \frac{3^n + a_n}{6^n} = \sum_{n=1}^{\infty} \frac{3^n}{6^n} + \sum_{n=1}^{\infty} \frac{a_n}{6^n} = \sum_{n=1}^{\infty} \underbrace{\left(\frac{1}{2}\right)^n}_{\text{geometric}} + 2 = 1+2 = 3$ (b)

7. a) divergence test
$\displaystyle \sum_{n=1}^{\infty} \frac{n}{n+1}$; $\displaystyle \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$
$\Rightarrow$ series diverges

(b) ratio test
$\displaystyle \sum_{n=1}^{\infty} \frac{n}{2^n}$: $\displaystyle L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left(\frac{n+1}{2^{n+1}}\right)\left(\frac{2^n}{n}\right)$
$\displaystyle = \lim_{n\to\infty} \left(\frac{n+1}{n}\right) \cdot \frac{1}{2} = 1/2 < 1 \Rightarrow$ converges

c) $\displaystyle \sum_{n=1}^{\infty} \frac{n!}{2^n}$: $\displaystyle L = \lim_{n\to\infty} \frac{\cancel{(n+1)!}^{(n+1)}}{\cancel{2^{n+1}}_{2}} \cdot \frac{\cancel{2^n}}{\cancel{n!}_{1}} = \infty > 1$ $\Rightarrow$ diverges
ratio test

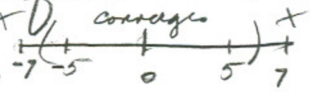
(d) $\displaystyle \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$ alt series test
① signs alternate
② $|a_{n+1}| < |a_n|$
③ $\lim_{n\to\infty} |a_n| = 0$
✓ converges

8. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5} = 1 - \frac{1}{2^5} + \frac{1}{3^5} - \cdots = 1 - .03125 + .004115 \bigg| - .000977$

$|\text{error}| < .001$

converges by alt series test $\doteq .972865$

9. If $\sum_{n=0}^{\infty} c_n x^n$ converges for $x=5$ and diverges for $x=-7$, then its interval of convergence is centered at 0 with radius of conv $5<R<7$.



(a) $\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} c_n \cdot 1^n \Rightarrow$ converges (b) $\sum_{n=0}^{\infty} c_n 4^n$ converges

(c) $\sum_{n=0}^{\infty} (-1)^n c_n 3^n = \sum_{n=0}^{\infty} c_n (-3)^n$ converges (d) $\sum_{n=0}^{\infty} c_n 8^n$ diverges

10. (a) $\sum_{n=1}^{\infty} \frac{n(x-2)^n}{3^n}$; ratio test $L = \lim_{n\to\infty} \left| \frac{(n+1)(x-2)^{n+1}}{3^{n+1}} \right| \left| \frac{3^n}{n(x-2)^n} \right| = \lim_{n\to\infty} \frac{n+1}{n} \frac{|x-2|}{3} = \frac{|x-2|}{3}$

the series converges if $L<1 \Rightarrow \frac{|x-2|}{3} < 1 \Rightarrow |x-2|<3 \Rightarrow -1<x<5 \Rightarrow R=3$.

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$: $L = \lim_{n\to\infty} \frac{|x|^{2(n+1)}}{(2(n+1))!} \cdot \frac{(2n)!}{|x|^{2n}} = \lim_{n\to\infty} \frac{|x|^2}{(2n+2)(2n+1)} = 0$ for all x

$\Rightarrow$ series converges for all real numbers $x \Rightarrow -\infty < x < \infty \Rightarrow R=\infty$.

11. (a) $1+x+x^2+\cdots$ is geometric with $a=1$, $r=x \Rightarrow$ it converges if $|r|<1 \Leftrightarrow |x|<1 \Leftrightarrow -1<x<1$.

(b) $\frac{1}{1+x} = \frac{1}{1-(-x)} = 1+(-x)+(-x)^2+(-x)^3+\cdots = 1-x+x^2-x^3+x^4-\cdots$. converges if $|(-x)|<1 \Leftrightarrow |x|<1 \Rightarrow R=1$.

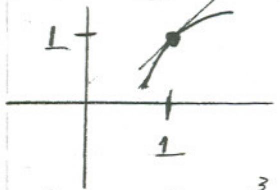
(c) $\ln(1+x) = \int \frac{1}{1+x}\,dx = \int (1-x+x^2-x^3+\cdots)\,dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + C_0$

(d) $\int_0^{.5} \frac{\ln(1+x)}{x}\,dx = \int_0^{.5} \frac{(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots)}{x}\,dx = \int_0^{.5} (1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots)\,dx$

$= x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} \cdots \Big|_0^{.5} = .5 - \frac{(.5)^2}{4} + \frac{(.5)^3}{9} \Big| - \frac{.5^4}{16} \doteq .45$

$.004$

12. $T(x) = \overset{f(1)}{1} + \overset{f'(1)}{1}(x-1) \overset{f''(1)/2}{-1}(x-1)^2$



slope $=1$
concave down

13. $f(x) = x^2+3x-4$ $f(0) = -4$; $f(2) = 6$
$f'(x) = 2x+3$ $f'(0) = 3$; $f'(2) = 7$
$f''(x) = 2$ $f''(0) = 2$; $f''(2) = 2$
$f'''(x) = 0$

Maclaurin $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$
$= -4 + 3x + 1x^2 =$ orig polynomial

Taylor: $f(x) = f(2) + f'(2)(x-2) + f''(2)\frac{(x-2)^2}{2} + \cdots$
$= 6 + 7(x-2) + 1(x-2)^2$

14. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$e^{-(x^2)} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!}$

$= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$