SM221, 4042, Examination 1

Spring Semester 2007

Conditions: You may use one page of notes for reference, and a calculator line the Voyage200 for computations. You may consult only with the instructor about questions on the examination.

Answer the question s on the paper provided, one question per page. Return the examination with your solutions to the problems.

Credit will be awarded for the solution of a problem—the lines of development leading to the answer-- not merely an answer to a problem. In cases of miscalculation, partial credit may be awarded based upon the extent of the development of an answer to a problem.

Notations: We represent points in 3-space in Cartesian (rectangular) coordinates by the ordered triple $P(x,y,z)$, in cylindrical coordinates by $P(r,\theta,z)$ and in spherical coordinates by $P(\rho,\theta,\phi)$.

1. While taking R&R on the Apollo 14 mission, then Capt. Alan Shepard tried a round of golf. Because of the Moon's smaller size and "mass concentrations", while in flight, with the origin of the coordinate system at the tee, the horizontal (x) axis under the line of flight of the ball, the vertical (y) axis up, and with the ***i*** and ***j*** unit vectors specified accordingly (see picture on board), assume that the acceleration that the ball experienced while in flight was

$$\mathbf{a}(t) = -\left(6+\cos(t)\right)\mathbf{j} \text{ ft/s}^2$$

He chipped the ball with a launch speed of about 100 ft/s, with a loft angle of about 30°. Taking the tee as the origin,

a) *(12 points)* What are the velocity and position vector functions governing the flight of the golf ball?
b) *(4 points)* On TV, the time of flight for the ball was clocked at about 16.6346 seconds. Does your model for the motion support this value, or was it as optical illusion? Justify your assertion, using your model.
c) *(4 points)* If the time of flight for the ball was 16.6346 seconds, how far downrange did the ball travel?
d) *(4 points)* What was its speed at impact?
e) *(6 points)* How far through the air did the ball travel along its trajectory?

Ed note: At the time, people were wow'ed by the video of the shot (still available on the web). Ironically, we've grown so jaded to the wonders and whimsy of the universe that hardly anyone took notice when a Russian cosmonaut launched a golf ball into orbit from the International Space Station in 2006.

Solution

a) We integrate the acceleration vector function with respect to time to obtain the general form for the velocity vector function. We use the initial velocity to compute the constants of integration, and to extract the velocity vector function specific for our problem.

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = -\int \left(6+\cos(t)\right)dt\,\mathbf{j}$$
$$\mathbf{v}(t) = -\left(6t+\sin(t)\right)\mathbf{j} + c_1\mathbf{i} + c_2\mathbf{j}$$
$$100\cos\left(\frac{\pi}{6}\right)\mathbf{i} + 100\sin\left(\frac{\pi}{6}\right)\mathbf{j} = \mathbf{v}(0) = -\left(6\cdot 0+\sin(0)\right)\mathbf{j} + c_1\mathbf{i} + c_2\mathbf{j}$$
$$50\sqrt{3}\,\mathbf{i} + 50\mathbf{j} = \mathbf{v}(0) = 0\mathbf{j} + c_1\mathbf{i} + c_2\mathbf{j}$$
$$c_1 = 50\sqrt{3};\ c_2 = 50$$
$$\mathbf{v}(t) = -\left(6t+\sin(t)\right)\mathbf{j} + 50\sqrt{3}\,\mathbf{i} + 50\mathbf{j}$$
$$\underline{\underline{\mathbf{v}(t)}} = \underline{\underline{50\sqrt{3}\,\mathbf{i} + \left(50-6t-\sin(t)\right)\mathbf{j}}}\ \text{ft/s}$$

We integrate the velocity vector function with respect to time to obtain the general form for the position vector function. We use the initial position to compute the constants of integration, and to extract the position vector function specific for our problem.

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int 50\sqrt{3}\,dt\,\mathbf{i} + \int \left(50-6t-\sin(t)\right)dt\,\mathbf{j}$$
$$\mathbf{r}(t) = 50\sqrt{3}\,t\,\mathbf{i} + \left(50t-3t^2+\cos(t)\right)\mathbf{j} + c_1\mathbf{i} + c_2\mathbf{j}$$
$$0\mathbf{i} + 0\mathbf{j} = \mathbf{r}(0) = 0\mathbf{i} + \left(0-0+\cos(0)\right)\mathbf{j} + c_1\mathbf{i} + c_2\mathbf{j}$$
$$0\,\mathbf{i} + 0\mathbf{j} = \mathbf{r}(0) = c_1\mathbf{i} + \left(1+c_2\right)\mathbf{j}$$
$$c_1 = 0;\ c_2 = -1$$
$$\mathbf{r}(t) = 50\sqrt{3}\,t\,\mathbf{i} + \left(50t-3t^2+\cos(t)\right)\mathbf{j} + 0\,\mathbf{i} - 1\mathbf{j}$$
$$\underline{\underline{\mathbf{r}(t)}} = \underline{\underline{50\sqrt{3}\,t\,\mathbf{i} + \left(\cos(t)-1+50t-3t^2\right)\mathbf{j}}}\quad \text{ft}$$

b) If the time of flight is $t_f = 16.6346\ \text{s}$, then $y\left(t_f\right)$ should be zero. We test this condition:

$$y\left(t_f\right) = y(16.6346) = \cos(16.6346) - 1 + 50\sqrt{3}(16.6346) - 3(16.6346)^2$$
$$\cong -0.0003\ \text{ft}$$

That's about ***right on*** impact.

c) If the time of flight is $t_f = 16.6346\ \text{s}$, the horizontal distance downrange that the golf ball travels is

$$x\left(t_f\right) = x(16.6346) = 50\sqrt{3}\,(16.4346) \cong 1440.6\ \text{ft (!!)}$$

d) We can compute the speed of impact by computing the magnitude of $\mathbf{v}\left(t_f\right)$. The quickest way to do that would be to compute the square root of the dot product of this vector with itself.

$$\mathbf{v}(t_f) = \mathbf{v}(16.6346) = 50\sqrt{3}\,\mathbf{i} + (50 - 6\cdot(16.6346) - \sin(16.6346))\,\mathbf{j}$$
$$\cong 86.603\,\mathbf{i} - 49.08\,\mathbf{j}$$
$$\|\mathbf{v}(t_f)\| \cong \sqrt{86.603^2 + (-49.008)^2} \cong 99.508 \text{ ft/s}$$

Ed note: The result is reasonable. Even though the variation in the acceleration is approximate, we anticipate that the gravitational force is *conservative*. Since the height of the launch point and the point of impact essentially coincide, we expect the kinetic energy of the ball at impact (hence, the speed of the ball) to coincide with the kinetic energy of the ball (hence, its speed) at launch.

e) To find the length of the arc the ball travels, we integrate the speed function over the duration of the flight. First, the speed function is the magnitude of the velocity vector function.

$$\|\mathbf{v}(t)\| = \sqrt{\left(50\sqrt{3}\right)^2 + (50 - 6t - \sin(t))^2}$$
$$L = \int_{t=0}^{t_f} \|\mathbf{v}(t)\|\,dt = \int_{t=0}^{16.6346} \sqrt{\left(50\sqrt{3}\right)^2 + (50 - 6t - \sin(t))^2}\,dt$$
$$L \cong 1516.56 \text{ ft}$$

Observe that the ball travels a greater distance than the downrange distance, but not too much. That's because the ball didn't "loft" that much.

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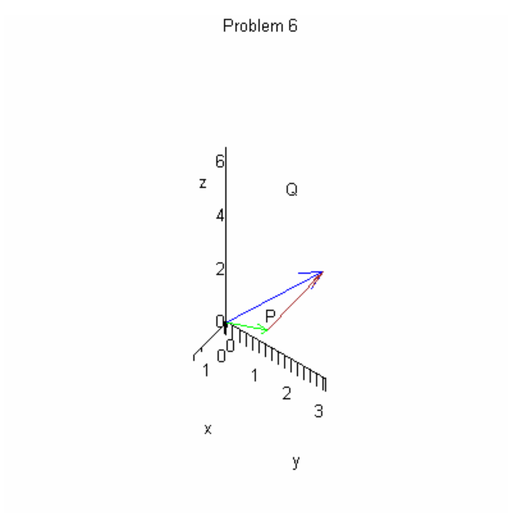
2. *(12 points)* From the origin $O(0,0,0)$ and the two points $P(1,2,2)$ and $Q(0,3,4)$, we can construct three vectors $\mathbf{a} = \overrightarrow{OP}$, $\mathbf{b} = \overrightarrow{OQ}$, and $\mathbf{c} = \overrightarrow{PQ}$, as shown in the figure:



a) Compute the angle $\angle POQ$ between the vectors $\mathbf{a}$ and $\mathbf{b}$.

b) Compute the scalar projection of vector $\mathbf{b}$ onto vector $\mathbf{a}$.

c) Compute the analytical specification of a vector perpendicular to the vectors $\mathbf{a}$ and $\mathbf{b}$.

Solution
First, we specify analytically the vectors **a** and **b**, as they are the only ones germane to the questions:

$$\mathbf{a} = \overrightarrow{OP} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$
$$\mathbf{b} = \overrightarrow{OQ} = 3\mathbf{j} + 4\mathbf{k}$$

a) Set $\theta = \angle POQ$. Then

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{ab}\,\boldsymbol{cos}(\theta)$$
$$6+8 = \sqrt{(1+4+4)}\sqrt{(9+16)}\,\boldsymbol{cos}(\theta)$$
$$\boldsymbol{cos}(\theta) = \frac{14}{\sqrt{9}\sqrt{25}} = \frac{14}{15}$$
$$\theta = \boldsymbol{cos}^{-1}\left(\frac{14}{15}\right) \cong 0.367\,(rad) \cong 21.04^{\circ}$$

b) The scalar projection is

$$Comp_{\mathbf{a}}\mathbf{b} = b\,\boldsymbol{cos}(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$$
$$= \frac{14}{\sqrt{9}} = \frac{14}{3} \cong 4.67$$

c) Since the cross product of the two vectors is perpendicular to each, we compute

$$\mathbf{a} \times \mathbf{b} = \boldsymbol{det}\left(\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 0 & 3 & 4 \end{bmatrix}\right) = (8-6)\mathbf{i} + (0-4)\mathbf{j} + (3-0)\mathbf{k}$$
$$= 2\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$$

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3. *(8 points)* Does the line

$$x(t) = t-3$$
$$y(t) = t-6$$
$$z(t) = t+3$$

Intersect the plane $x - 2y + z = 9$?

If so, specify the coordinates of the point of intersection; if not, briefly justify why not?

Solution
Alternative 1.
The equations for the line and the equation for the plane constitutes a system of four equations for four unknowns, (x, y, z, t). If the equations are *consistent* there will be a unique solution (a quartet of values) for these variables that will determine when and where the motion whose "vapor trail" constitutes the line

intersects the plane. If the equations are *inconsistent*, a contradiction will arise as we try to solve for the four variable; in which case, the line does not intersect the plane.

To solve the equations, let us substitute the first three equations into the fourth to obtain one equation for the one unknown, t:

$$\begin{aligned} x-2y+z&=9 \\ (t-3)-2(t-6)+(t+3)&=9 \\ (2-2)t+(-3+12+3)&=9 \\ 0t&=3 \\ 0&=3 \ \#\# \end{aligned}$$

We obtain a contradiction. We conclude that the line ***does not intersect*** the plane.

Alternative 2

We observe that the "launch point" for the motion (the $t=0$ location) is not a point on the plane:

$$\begin{aligned} (x(0),y(0),z(0))&=(-3,-6,3) \\ x(0)-2y(0)+z(0)&\overset{?}{=}9 \\ (-3)-2(-6)+3&\overset{?}{=}9 \\ -15+3&\overset{?}{=}9 \\ -12&\neq 9 \end{aligned}$$

The line will intersect the plane only if the (velocity) vector tangent to the motion that delineates the line is *not* perpendicular to the normal vector to the plane. We extract from the parametric equations for the straight line motion the (velocity vector), we extract from the equation for the plane the components of a vector normal to the plane, and we compute the dot product of the two. If the dot product is zero, the two vectors are perpendicular, and the line is parallel to the plane.

The components of the (velocity) vector for the straight line motion we identify from the coefficients of the parameter in the parametric equations for the line:

$$\left.\begin{aligned} x(t)&=t-3 \\ y(t)&=t-6 \\ z(t)&=t+3 \end{aligned}\right\}\Rightarrow \mathbf{v}(t)=1\mathbf{i}+1\mathbf{j}+1\mathbf{k}$$

The components for a normal vector to the plane we extract from the coefficients of the variables in the equation for the plane.

$$x-2y+z=9$$
$$1(x-9)-2(y-0)+1(z-0)=0\Rightarrow \mathbf{N}=1\mathbf{i}-2\mathbf{j}+1\mathbf{k}$$

The dot product of the two vectors under consideration is immediate:

$$\mathbf{N}\bullet\mathbf{v}=\left(1\mathbf{i}-2\mathbf{j}+1\mathbf{k}\right)\bullet\left(1\mathbf{i}+1\mathbf{j}+1\mathbf{k}\right)=1-2+1=0$$

The dot product is zero; the two vectors are perpendicular. The line either lies entirely in the plane or it lies parallel to the plane, never to intersect. Since the “launch point” does not lie in the plane, the latter alternative is the case. The line ***does not intersect*** the plane.

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4. *(10 points)* Construct the equation for a plane that *intersects* the line of problem #3 and is perpendicular to it.

Solution
If we take the (velocity) vector for the motion specified by the parametric equations that produce the line, to be the normal vector to the plane we desire, then *by design* the plane will be perpendicular to the line. In addition, we need an “anchor point” in addition to a normal vector to specify a plane. Let’s take the anchor point to be the “launch point” (the $t=0$ point) in the straight line motion.

The components of the (velocity) vector for the straight line motion we identify from the coefficients of the parameter in the parametric equations for the line:

$$\left.\begin{array}{l} x(t)=t-3 \\ y(t)=t-6 \\ z(t)=t+3 \end{array}\right\}\Rightarrow \mathbf{v}=1\mathbf{i}+1\mathbf{j}+1\mathbf{k}$$

The coordinates for the launch point are

$$P_0\left(x(0),y(0),z(0)\right)=(-3,-6,3)$$

The equation for the plane now follows from the following line of reasoning. Let $R(x,y,z)$ denote a generic point on the plane. Then the vector $\overrightarrow{P_0R}$ that specifies the position of the generic point relative to the anchor point for the plane must be perpendicular to the normal vector $\mathbf{N}$ to the plane. If we set that normal vector to be the velocity vector for the line, $\mathbf{N}=\mathbf{v}$, the requirement that the normal and relative position vectors be perpendicular produces the equation of the plane we desire.

$$\begin{aligned} \mathbf{N}=\mathbf{v}&=1\mathbf{i}+1\mathbf{j}+1\mathbf{k} \\ \overrightarrow{P_0R}&=\left(x-(-3)\right)\mathbf{i}+\left(y-(-3)\right)\mathbf{j}+(z-6)\mathbf{k}=(x+3)\mathbf{i}+(y+3)\mathbf{j}+(z-6 \\ \mathbf{v}\bullet\overrightarrow{P_0R}&=0 \quad (\text{require } \perp) \\ (1\mathbf{i}+1\mathbf{j}+1\mathbf{k})\bullet\left((x+3)\mathbf{i}+(y+3)\mathbf{j}+(z-6)\mathbf{k}\right)&=0 \\ (x+3)+(y+3)+(z-6)&=0 \\ x+y+z&=0 \end{aligned}$$

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5. *(12 points)* Describe and sketch the following geometric objects:

a) $x^2+y^2-z+1=0$,

b) $y^2 + z = 5$,

c) the intersection of a) and b).

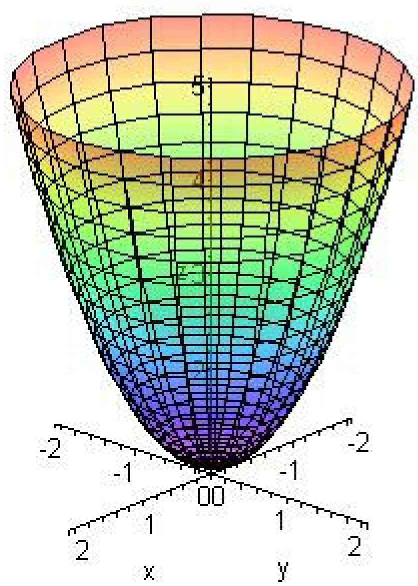
Solution

a) Represent the equation as

$$(z-1) = x^2 + y^2$$

Taking $z = c \left(c \geq 1\right)$, a constant, we see that "horizontal slices" of the graph of the equation produce circles. Setting $x^2 + y^2 = r^2$, we see that "vertical slices" in the (r, z) plane (θ fixed) of cylindrical coordinates produce parabolae. Consequently, our surface must be a ***circular paraboloid***, aligned along the z axis ("odd exponent out"), opening "upward" (positive z axis) from the vertex $(0,0,1)$. Its shape is what we'd expect, which we draw using Maple for convenience:

Problem 5, x^2 + y^2 - z + 1 = 0

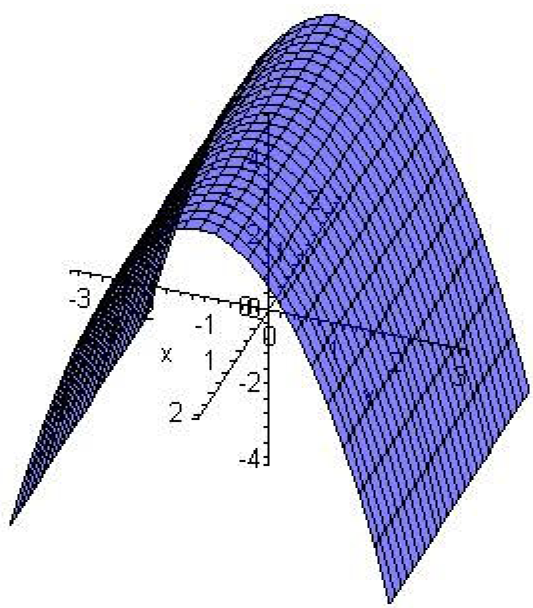


b) Represent the equation as

$$z = 5 - y^2$$

Since the *x* coordinate is absent from the equation, the surface must be a *cylinder* consisting of straight lines parallel to the *x* axis. The "ribs" or generating rays along which these lines are affixed is specified by the equation above: a parabola in the *yz* plane that is aligned along the z axis, opening *downward* (negative *z* axis) from the vertex (0,0,5). Consequently, our surface must be a ***parabolic cylinder***, aligned along the *x* axis ("missing coordinate"), opening "downward" (negative z axis) from the vertex $(0,0,5)$. Its shape is what we'd expect, which we draw using Maple for convenience:

Problem 5, y^2 - z = 5



c) If we add the equations for the two surfaces, we eliminate the *z* variable; hence, we gain the equation for the "shadow" of the intersection of the two surfaces that we'd see in the *xy* plane, were we to look straight down along the *z* axis. The sum of the two equations is

$$x^2 + 2y^2 = 4$$
$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{\sqrt{2}}\right)^2 = 1$$

The equation characterizes an ellipse, with center on the z-axis, "x-radius" of 2, "y-radius" of $\sqrt{2}$. In particular, the intersection of the two surfaces is a *closed curve*.

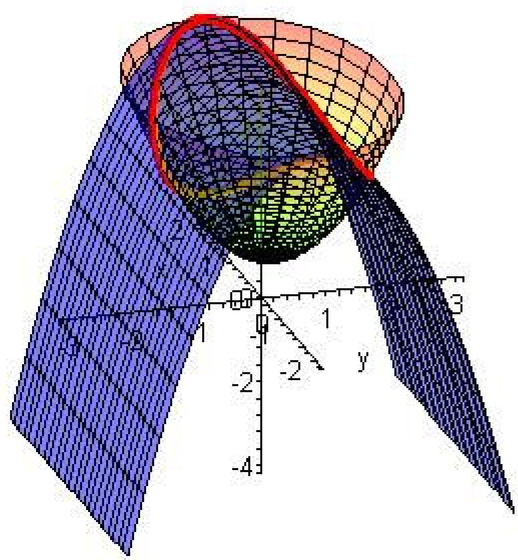
If we subtract the equations for the two surfaces, we eliminate the *y* variable; hence, we gain the equation for the "shadow" of the intersection of the two surfaces that we'd see in the *xz* plane, were we to look straight along the *y* axis. The difference between the two equations is

$$x^2 - 2z = -6$$
$$z = 3 + \frac{1}{2}x^2$$

The equation characterizes a parabola in the xz plane, vertex $(0,3)$ opening upward along the positive z axis. The requirement from the previous equation (of the ellipse) inform us that x cannot exceed the value 2.

We conclude that the intersection is a "wavy" closed curve that's elliptical and parabolic in shape. Indeed intersecting the two surfaces displays such a curve, drawn using Maple for convenience.

Prob.5 intersection



6. *(12 points)* Does the motion

$$\mathbf{r}(t) = 2\cos(t)\,\mathbf{i} + \sqrt{2}\sin(t)\,\mathbf{j} + \left(4 + \cos^2(t) - \sin^2(t)\right)\mathbf{k}$$

a) traverse the curve described in problem #5c)? If so, demonstrate why; if not, demonstrate why not.
b) What is its velocity vector function?
c) What is its acceleration?

Solution
a) To traverse the curve that is the intersection, the coordinate functions would have to satisfy simultaneously the equations for both surfaces. We try the equation for the circular paraboloid first.

$$\text{For } x(t)=2\cos(t),\ y(t)=\sqrt{2}\sin(t),\ z(t)=4+\cos^2(t)-\sin^2(t)$$

$$x(t)^2+y(t)^2-z(t)+1\overset{?}{=}0$$

$$4\cos^2(t)+2\sin^2(t)-\left(4+\cos^2(t)-\sin^2(t)\right)+1\overset{?}{=}0$$

$$3\cos^2(t)+3\sin^2(t)-4+1\overset{?}{=}0$$

$$3-4+1\overset{?}{=}0$$

$$0=0 \ \text{ Yes!}$$

The trajectory lies on the circular paraboloid. Now we examine the parabolic cylinder:

$$\text{For } x(t)=2\cos(t),\ y(t)=\sqrt{2}\sin(t),\ z(t)=4+\cos^2(t)-\sin^2(t)$$

$$y(t)^2+z(t)\overset{?}{=}5$$

$$2\sin^2(t)+\left(4+\cos^2(t)-\sin^2(t)\right)\overset{?}{=}5$$

$$\cos^2(t)+\sin^2(t)+4\overset{?}{=}5$$

$$1+4\overset{?}{=}5$$

$$5=5 \ \text{ Yes!}$$

The trajectory lies on the parabolic cylinder also. Consequently, the trajectory for the motion ***is the curve of intersection*** of the two surfaces.

b) We compute the velocity vector function by differentiating the position vector function.

$$\begin{aligned}\mathbf{v}(t)=\frac{d\mathbf{r}(t)}{dt}&=\frac{d\left(2\cos(t)\right)}{dt}\mathbf{i}+\frac{d\left(\sqrt{2}\sin(t)\right)}{dt}\mathbf{j}+\frac{d\left(4+\cos^2(t)-\sin^2(t)\right)}{dt}\mathbf{k}\\&=-2\sin(t)\mathbf{i}+\sqrt{2}\cos(t)\mathbf{j}+\left(-2\cos(t)\sin(t)-2\sin(t)\cos(t)\right)\mathbf{k}\\&=-2\sin(t)\mathbf{i}+\sqrt{2}\cos(t)\mathbf{j}-4\cos(t)\sin(t)\mathbf{k}\end{aligned}$$

c) We compute the acceleration vector function by differentiating the velocity vector function.

$$\begin{aligned}\mathbf{a}(t)=\frac{d\mathbf{v}(t)}{dt}&=\frac{d\left(-2\sin(t)\right)}{dt}\mathbf{i}+\frac{d\left(\sqrt{2}\cos(t)\right)}{dt}\mathbf{j}-4\frac{d\left(\cos(t)\sin(t)\right)}{dt}\mathbf{k}\\&=-2\cos(t)\mathbf{i}-\sqrt{2}\sin(t)\mathbf{j}-4\left(-\sin^2(t)+\cos^2(t)\right)\mathbf{k}\\&=-2\cos(t)\mathbf{i}-\sqrt{2}\sin(t)\mathbf{j}-4\left(\cos^2(t)-\sin^2(t)\right)\mathbf{k}\end{aligned}$$

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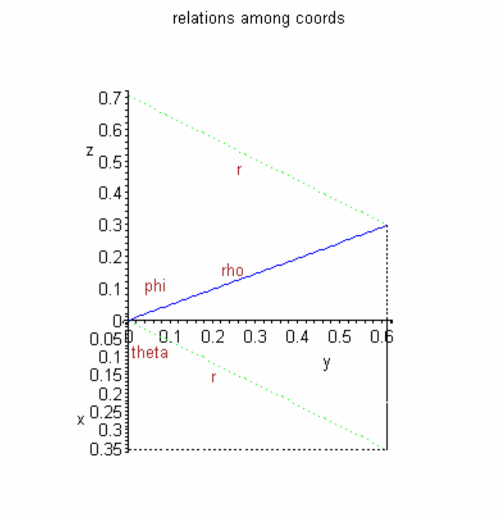
7. *(16 points)* Sketch (including a coordinate system) or describe in words the surfaces represented by the following equations. If describing the surface in words, be specific about the location, orientation and shape of the surface. ***Hint***: you may represent the equation in any coordinate system that is most comfortable for you.

a) $r^2 + z^2 = 4$
b) $r^2 - z^2 = 4$
c) $\rho\cos(\phi) = 4$
d) $r\cos(\theta) = 4$

Solution

a) If we draw the figure that relates spherical, cylindrical and Cartesian coordinates,



We see the relation between the cylindrical radial coordinate *r* and the Cartesian coordinates *x, y* is the Pythagorean relationship

$$\boxed{r^2 = x^2 + y^2}$$

Consequently, in Cartesian coordinates, the equation is

$$x^2 + y^2 + z^2 = 4$$

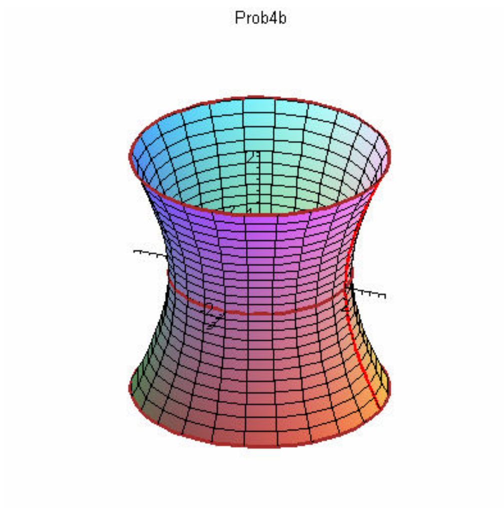
Pro forma, we recognize this equation as characterizing a ***sphere***, center (0,0,0) and radius 2. Alternatively, we could deduce this surface by taking slices of it and producing a wireframe.

b) Using the relation between the cylindrical radial coordinate *r* and the Cartesian coordinates *x, y*, the equation becomes, in Cartesian coordinates,

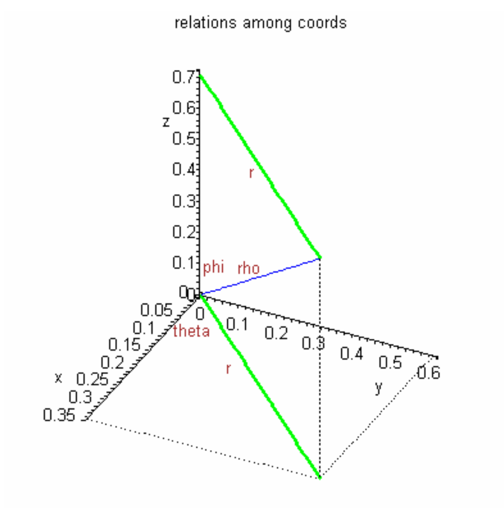
$$x^2 + y^2 - z^2 = 4$$

Pro forma, we recognize this equation as characterizing a ***one-sheeted circular hyperboloid*** aligned along and symmetric about the z axis. This "wormhole" has its narrowest constriction in the $z = 0$ plane, where the circle of intersection has radius 2.

Alternatively, we could deduce this surface by taking slices of it and producing a wireframe. Slicing with planes parallel to the *z* axis produces circles aligned along the *z* axis, whose radii increase as you move away from the xy plane. Slicing with the $x = 0$ plane (yz plane) produces the hyperbola that serve as the "backbone" to which the circles attach.



c) If we draw the figure that relates spherical, cylindrical and Cartesian coordinates,



we can deduce (or recall) the relation

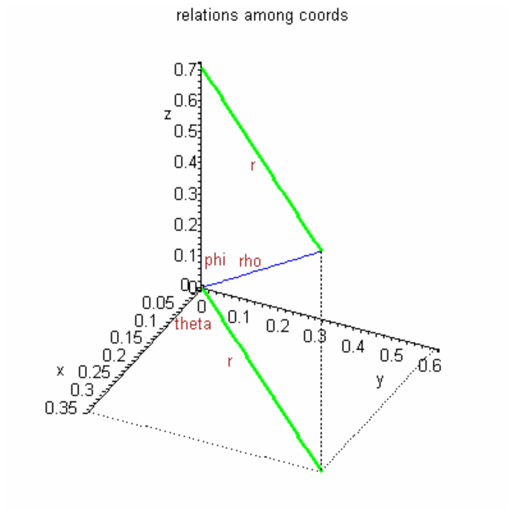
$$z = \rho \boldsymbol{cos}(\phi)$$

Consequently, in cylindrical or Cartesian coordinates, the equation for the surface reads as

$$\boxed{z = 4}$$

Perhaps you recognize immediately that this equation specifies a ***plane*** parallel to the *xy* $(\text{or } z = 0)$ plane and four "units" above it.

d) If we draw the figure that relates spherical, cylindrical and Cartesian coordinates,



we can deduce (or recall) the relation

$$x = r\,\boldsymbol{cos}(\theta)$$

Consequently, Cartesian coordinates, the equation for the surface reads as

$$\boxed{x = 4}$$

Perhaps you recognize immediately that this equation specifies a ***plane*** parallel to the *yz* $(\text{or } x = 0)$ plane and four "units" in front of it.

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