SM221 4042
Spring Semester 2007
Examination 2
Solutions

Conditions: You may use symbolic calculators, such as the TI-92 or Vantage 200 series, to assist you in your computations.

1. *(15 points)* Compute $\frac{\partial g}{\partial u}(u,v)$, if $g(u,v) = f(x(u,v), y(u,v))$, and

$$f(x,y) = x\sin(2y)$$
$$x(u,v) = \ln(u+v)$$
$$y(u,v) = e^{uv}$$

Solution

1. Erect the chain rule

By the tree diagram

$$g(u,v) \approx \left\{ \begin{array}{ccccccccc} & & & & f & & & & \\ & & f_x & / & & \backslash & f_y & & \\ & & x & & & & y & & \\ x_u & / & & \backslash & x_v & / & y_u & \backslash & y_v \\ & u & & v & & u & & v & \end{array} \right.$$

We deduce that the chain rule for computing $g_u(u,v)$ should be

$$g_u(u,v) = \left(f_x(x,y)\right)\Big|_{\substack{x=x(u,v)\\ y=y(u,v)}} \cdot \left(x_u(u,v)\right) + \left(f_y(x,y)\right)\Big|_{\substack{x=x(u,v)\\ y=y(u,v)}} \cdot \left(y_u(u,v)\right)$$

where

$$f(x,y) = x\sin(2y)$$
$$x(u,v) = \ln(u+v)$$
$$y(u,v) = e^{uv}$$

2. Compute the pieces

We compute the derivatives of the various links in the chain and evaluate them at their appropriate places.

$$f_x(x,y)=\sin(2y) \quad \left(f_x(x,y)\right)\Big|_{\substack{x=\ln(u+v)\\ y=e^{uv}}}=\sin\left(2e^{uv}\right)$$

$$f_y(x,y)=2x\cos(2y) \quad \left(f_y(x,y)\right)\Big|_{\substack{x=\ln(u+v)\\ y=e^{uv}}}=2\ln(u+v)\cos\left(2e^{uv}\right)$$

$$x_u(u,v)=\frac{1}{u+v}$$

$$y_u(u,v)=ve^{uv}$$

3. Assemble the chain

We obtain

$$\begin{aligned} g_u(u,v) &= \left(f_x(x,y)\right)\Big|_{\substack{x=x(u,v)\\ y=y(u,v)}}\cdot\left(x_u(u,v)\right)+\left(f_y(x,y)\right)\Big|_{\substack{x=x(u,v)\\ y=y(u,v)}}\cdot\left(y_u(u,v)\right) \\ &= \sin\left(2e^{uv}\right)\left(\frac{1}{u+v}\right)+2\ln(u+v)\cos\left(2e^{uv}\right)\left(ve^{uv}\right) \\ &= \frac{\sin\left(2e^{uv}\right)}{u+v}+2ve^{uv}\ln(u+v)\cos\left(2e^{uv}\right) \end{aligned}$$

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2. *(20 points)*

a) Suppose $f(x,y)=x\cos\left(y^2\right)$, $x=x(t)$, $y=y(t)$, and $g(t)=f(x(t),y(t))$.

Find $\frac{dg}{dt}(1)=\frac{dg}{dt}(t)_{|t=1}$, if

$x(0)=0$	$x(1)=0.524$	$x(2)=0.866$
$y(0)=0$	$y(1)=0.8863$	$y(2)=1.047$
$\frac{dx}{dt}(0)=1.047$	$\frac{dx}{dt}(1)=0.866$	$\frac{dx}{dt}(2)=-0.524$
$\frac{dy}{dt}(0)=0.524$	$\frac{dy}{dt}(1)=0.524$	$\frac{dy}{dt}(2)=0.524$

b) Forge the formal chain rule with which you'd compute $\frac{\partial L}{\partial u}(u,v)$, if the function $L=f(x,y,z)$, where $x=x(t),\ y=y(t),\ z=z(t)$, and where $t=t(u,v)$. **DO NOT** attempt to execute the chain rule; rather, just set it up.

Solution

a)

1. Create the architecture of the chain

By the tree diagram

$$g(t) \approx \left\{ \begin{array}{ccccc} & & f & & \\ & f_x \quad / & & \backslash \quad f_y & \\ & x & & y & \\ x_t & | & & | & y_t \\ & t & & t & \end{array} \right.$$

We deduce that the chain rule for computing $g_t(t)$ should be

$$g_t(t) = \left(f_x(x,y)\right)\Big|_{\substack{x=x(t)\\ y=y(t)}} \cdot \left(x_t(t)\right) + \left(f_y(x,y)\right)\Big|_{\substack{x=x(t)\\ y=y(t)}} \cdot \left(y_t(t)\right)$$

where

$$f(x,y) = x\cos\left(y^2\right)$$

In particular, at $t=1$, we have the chain rule

$$\left(g_t(t)\right)\Big|_{t=1} = \left(f_x(x,y)\right)\Big|_{\substack{x=x(1)\\ y=y(1)}} \cdot \left(x_t(1)\right) + \left(f_y(x,y)\right)\Big|_{\substack{x=x(1)\\ y=y(1)}} \cdot \left(y_t(1)\right)$$

2. Evaluate the pieces at their appropriate places
We obtain by computation

$$f_x(x,y) = \cos\left(y^2\right) \quad f_x(x,y)\Big|_{\substack{x=x(1)=0.524\\ y=y(1)=0.8863}} = \cos\left((0.8863)^2\right) \cong 0.707$$

$$f_y(x,y) = -2xy\sin\left(y^2\right) \quad f_y(x,y)\Big|_{\substack{x=x(1)=0.524\\ y=y(1)=0.8863}} = -2(0.524)(0.8863)\sin\left((0.8863)^2\right) \cong -0.656$$

We obtain from the table

$$x_t(1) = 0.866 \quad y_t(1) = 0.524$$

3. Assemble the pieces
By the chain rule at $t=1$, we obtain

$$\boxed{\begin{aligned}\left(g_t(t)\right)\Big|_{t=1} &= \left(f_x(x,y)\right)\Big|_{\substack{x=x(1)\\ y=y(1)}} \cdot \left(x_t(1)\right) + \left(f_y(x,y)\right)\Big|_{\substack{x=x(1)\\ y=y(1)}} \cdot \left(y_t(1)\right) \\ &\cong 0.707 \cdot 0.866 - 0.656 \cdot 0.524 = 0.2685\end{aligned}}$$

b)
By the tree diagram

$$
L(u,v)\approx\left\{\begin{array}{ccccccccccccc}
 & & & & & & f & & & & & & \\
 & & & / & f_x & & | & f_y & & \backslash & f_z & & \\
 & & x & & & & y & & & & z & & \\
 & x_t & | & & & & | & & & & | & z_t & \\
 & & t & & & & t & & & & t & & \\
t_u & / & & \backslash & t_v \quad t_u & / & & \backslash & t_v \quad t_u & / & & \backslash & t_v \\
u & & & & v \quad u & & & & v \quad u & & & & v
\end{array}\right.
$$

We deduce the chain rule (recognizing the requirement to evaluate each component at its appropriate place)

$$
\boxed{\begin{aligned}
L_u(u,v) = {} & f_x(x,y,z)\Big|_{\substack{x=x(t(u,v))\\ y=y(t(u,v))\\ z=z(t(u,v))}} \cdot x_t(t)\Big|_{t=t(u,v)} \cdot t_u(u,v) + \\
& f_y(x,y,z)\Big|_{\substack{x=x(t(u,v))\\ y=y(t(u,v))\\ z=z(t(u,v))}} \cdot y_t(t)\Big|_{t=t(u,v)} \cdot t_u(u,v) + \\
& f_z(x,y,z)\Big|_{\substack{x=x(t(u,v))\\ y=y(t(u,v))\\ z=z(t(u,v))}} \cdot z_t(t)\Big|_{t=t(u,v)} \cdot t_u(u,v) +
\end{aligned}}
$$

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3. *(30 points)* Given the function

$$F(x,y)=\sqrt{x^2-y^2}$$

a) Build the algebraic equation for and sketch the level set containing the point $P(5,4)$.

b) Evaluate the gradient of $F(x,y)$ at the point $P(5,4)$ and position the vector at the field point on the sketch of the level set of part a.

c) If you move from the field point towards the point $Q(6,6)$, what is the derivative of the function in that direction?

d) In what direction (**unit vector**) must you move in order for the function to increase in value as rapidly as possible? What is that maximum rate of increase?

e) What is the equation of the line tangent to the level curve for $F(x,y)$ at the point $P(5,4)$?

Solution

a) The algebraic equation for the level curve for a function of two variables is

$$F(x,y)=c$$

where the level value is determined from given information. Here, we infer that the point $P(5,4)$ is a point on the curve. Consequently,

$$c=F(5,4)=\sqrt{5^2-4^2}=\sqrt{9}=3$$

The equation for the level curve of the function through the point $P(5,4)$ is

$$\boxed{\sqrt{x^2 - y^2} = 3}$$

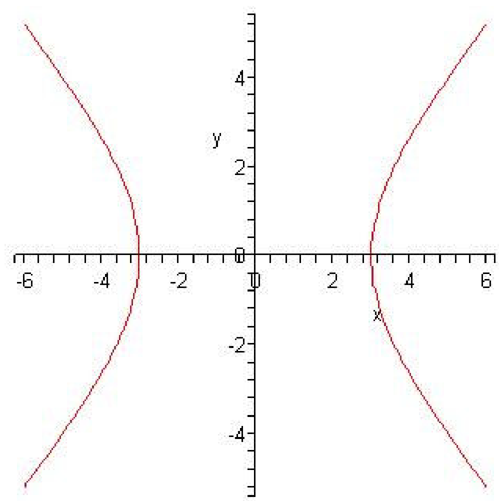
When we square the equation, we can recognize the nature of the curve immediately:

$$x^2 - y^2 = 9,$$

which is ***a hyperbola*** with two branches, symmetric and crossing the x-axis at $(-3,0)$ and $(3,0)$, symmetric and avoiding (not crossing) the y-axis. The ***right-branch*** (in the first and fourth quadrants) is the branch that contains the point $P(5,4)$ and corresponds to the level curve of interest. The left-branch is the level curve

$$\sqrt{x^2 - y^2} = -3.$$

A picture of the level curve is



b) The gradient vector field for the function $F(x, y)$ is

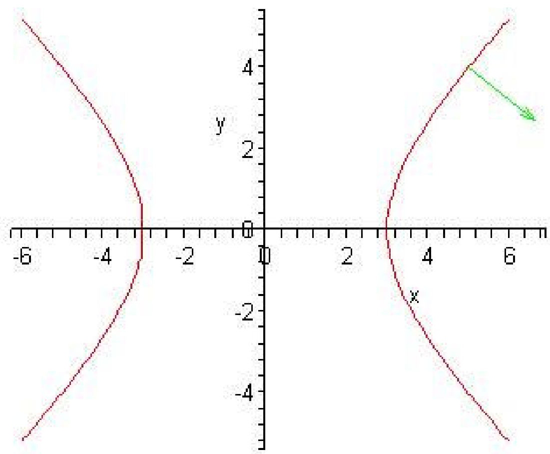
$$\begin{aligned}\nabla F(x, y) &= F_x(x, y)\mathbf{i} + F_y(x, y)\mathbf{j} \\ &= \frac{1}{2}\frac{2x}{\sqrt{x^2 - y^2}}\mathbf{i} - \frac{1}{2}\frac{2y}{\sqrt{x^2 - y^2}}\mathbf{j} \\ &= \frac{x}{\sqrt{x^2 - y^2}}\mathbf{i} - \frac{y}{\sqrt{x^2 - y^2}}\mathbf{j}\end{aligned}$$

When we evaluate it at the field point $P(5,4)$, we obtain the vector

$$\boxed{\nabla F(5,4) = \frac{5}{\sqrt{5^2 - 4^2}}\mathbf{i} - \frac{4}{\sqrt{5^2 - 4^2}}\mathbf{j} = \frac{5}{3}\mathbf{i} - \frac{4}{3}\mathbf{j}}$$

When this vector is superimposed on the level curve at the point $P(5,4)$, we obtain the graphic

Prob3, level curve with gradf(5,4)



Observe that the gradient is orthogonal to the level curve at the point, and that it points "to the right and down", as we require from its analytic specification.

c) If you move from the point $P(5,4)$ to the point $Q(6,6)$, you are moving along the vector

$$\overrightarrow{PQ} = (6-5)\mathbf{i} + (6-4)\mathbf{j} = \mathbf{i} + 2\mathbf{j}$$

Its direction is specified by the unit vector

$$\mathbf{u} = \frac{\overrightarrow{PQ}}{\left\|\overrightarrow{PQ}\right\|} = \frac{\mathbf{i} + 2\mathbf{j}}{\sqrt{1^2 + 2^2}} = \frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}$$

We recognize that our function $F(x,y) = \sqrt{x^2 - y^2}$ at all points (x,y) for which $x^2 > y^2$. In particular, our point of interest, $P(5,4)$, satisfies this condition. Consequently, as we move off the point $P(5,4)$ in the direction $\mathbf{u}$, the directional derivative for $F(x,y)$ is

$$
\begin{aligned}
D_{\mathbf{u}}F(5,4) &= \nabla F(5,4)\cdot\mathbf{u} \\
&= \left(\frac{5}{3}\mathbf{i}-\frac{4}{3}\mathbf{j}\right)\cdot\left(\frac{1}{\sqrt{5}}\mathbf{i}+\frac{2}{\sqrt{5}}\mathbf{j}\right)=\frac{\sqrt{5}}{3}-\frac{8}{3\sqrt{5}} \\
&= \frac{5\sqrt{5}}{15}-\frac{8\sqrt{5}}{15}=-\frac{\sqrt{5}}{5}
\end{aligned}
$$

d) By theorem, if the function of interest is differentiable at the point of interest, as $F(x,y)$ is at the point $P(5,4)$, then the maximum rate of change is

$$D_{\mathbf{u}_{\max}}F(5,4)=\left\|\nabla F(5,4)\right\|$$

and the direction in which this maximum rate will be attained is

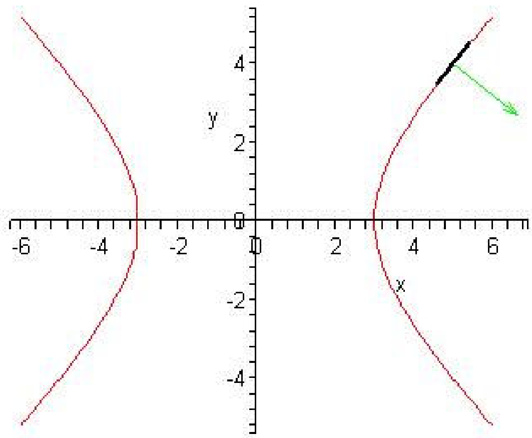
$$\mathbf{u}_{\max}=\frac{\nabla F(5,4)}{\left\|\nabla F(5,4)\right\|}$$

We compute these quantities:

$$
\begin{aligned}
D_{\mathbf{u}_{\max}}F(5,4) &= \left\|\nabla F(5,4)\right\| = \sqrt{\left(\frac{5}{3}\right)^2+\left(\frac{4}{3}\right)^2}=\frac{\sqrt{41}}{3} \\
\mathbf{u}_{\max} &= \frac{\nabla F(5,4)}{\left\|\nabla F(5,4)\right\|}=\frac{\left(\frac{5}{3}\mathbf{i}-\frac{4}{3}\mathbf{j}\right)}{\left(\frac{\sqrt{41}}{3}\right)}=\frac{5}{\sqrt{41}}\mathbf{i}-\frac{4}{\sqrt{41}}\mathbf{j}
\end{aligned}
$$

e) If we draw through $P(5,4)$ the tangent line to the level curve, we obtain the figure

Prob3, level curve, gradf(5,4),and tangent at (5,4)



and we see that the tangent line is perpendicular to the gradient vector $\nabla F(5,4)$ at $P(5,4)$. If we choose a generic point $R(x,y)$ on the tangent line, the attribute of perpendicularity imposes the mathematical requirement

$$\nabla F(5,4) \cdot \overrightarrow{PR} = 0$$

where $\overrightarrow{PR}$ is the vector that specifies the position of the point $R(x,y)$, relative to the point $P(5,4)$. Its analytic specification is

$$\overrightarrow{PR} = (x-5)\mathbf{i} + (y-4)\mathbf{j}$$

When we expand the requirement of the dot product, we obtain the equation for the tangent line:

$$\boxed{\begin{aligned} \nabla F(5,4) \cdot \overrightarrow{PR} &= 0 \\ \left(\frac{5}{3}\mathbf{i} - \frac{4}{3}\mathbf{j}\right) \cdot \left((x-5)\mathbf{i} + (y-4)\mathbf{j}\right) &= 0 \\ \frac{5}{3}(x-5) - \frac{4}{3}(y-4) &= 0 \\ 5(x-5) - 4(y-4) &= 0 \\ 5x - 4y &= 9 \end{aligned}}$$

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4. *(15 points)* The following table samples the value of a function $f(x, y)$ in the vicinity of the point $(x, y) = (2,1)$:

		y		
		0.9	1.0	1.1
x	1.9	4.34	4.61	4.94
	2.0	4.73	5.00	5.33
	2.1	5.14	5.41	5.74

Estimate the directional derivative of $f(x, y)$ at the point $(2,1)$ in the direction of $(1.9,1.1)$.

Solution

We estimate the value of the directional derivative by stepping from the point $(2,1)$ to the point $(1.9,1.1)$ and computing the ratio of the change in the value of the function to the distance we've traversed:

$$D_{\mathbf{u}} f(2,1) \cong \frac{f(1.9,1.1) - f(2,1)}{\sqrt{(1.9-2.0)^2 + (1.1-1.0)^2}} = \frac{4.94-5.00}{\sqrt{2 \cdot (0.1)^2}} = \frac{-0.06}{0.1\sqrt{2}} \cong -0.424$$

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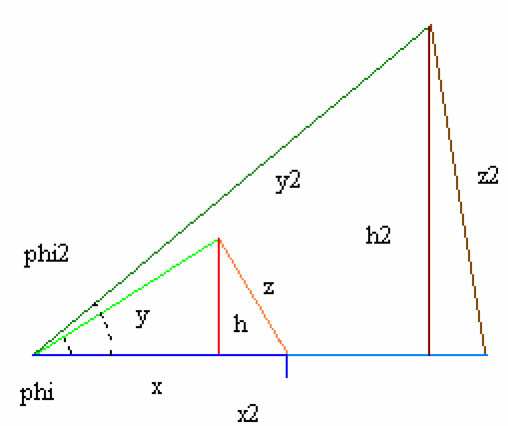
5. *(20 points)* The area of any triangle is given by the time honored formula $A = bh/2$, where b is the length of the base of the triangle and h is its altitude or height. The relation between the sides of a general triangle, its altitude and the angle included between two side is shown by the figure



When the base x of the triangle has length 40 in, when the side y above the base has length 50 in, and the contained angle ϕ is $\pi/6$ radians, suppose that the length of the base x is *increasing* as a rate of 3 in/sec, the contained angle ϕ is increasing at a rate of 0.05 radians/sec, and the length y of the side opposite the base is *increasing* at the rate of 2 in/sec.

a) What's the rate of change of the altitude (height) of the triangle?
b) What's the rate of change of the area of the triangle?

Solution

Compile the problem

Variables

For this problem, our variables are:

$t =$ time (sec), an independent variable
$x =$ the base of the triangle, a variable that depends upon t
$y =$ the side of the triangle that encloses the angle, a variable that depends upon t
$\phi =$ the enclosed angle, a variable that depends upon t
$h =$ altitude, a variable that depends ultimately upon t, but through intermediate variables
$A =$ area of the triangle, a variable that depends ultimately upon t, but through intermediate variables

Algebraic Relations

From the figure and the given information, we have

$$h = h(t) = y(t)\sin(\phi(t))$$
$$A = A(t) = \frac{1}{2}x(t)h(t)$$

Formulate what we seek:

For the instant of time t_0 when

$$x(t_0) = 40\ in \qquad x'(t_0) = 3\ in/s$$
$$y(t_0) = 50\ in \qquad y'(t_0) = 2\ in/s$$
$$\phi(t_0) = \pi/6\ rad \qquad \phi'(t_0) = 0.05\ rad/s$$

a) Compute

$$\left.\frac{dh(t)}{dt}\right|_{t=t_0}$$

using product and/or chain rules.

b) Compute

$$\left.\frac{dA(t)}{dt}\right|_{t=t_0}$$

using product and/or chain rules.

Resolve the problem

a) Given the specification for $h(t)$ in the relations, we have

$$\left.\frac{dh}{dt}(t)\right|_{t=t_0} = \left.\left(\frac{d}{dt}\left(y(t)\sin\left(\phi(t)\right)\right)\right)\right|_{t=t_0}$$
$$= y'(t_0)\sin\left(\phi(t_0)\right) + y(t_0)\cos\left(\phi(t_0)\right)\phi'(t_0)$$

Using the specifications from the given information, we have

$$\left.\frac{dh}{dt}(t)\right|_{t=t_0} = y'(t_0)\sin\left(\phi(t_0)\right) + y(t_0)\cos\left(\phi(t_0)\right)\phi'(t_0)$$
$$= 2\sin(\pi/6) + 50\cos(\pi/6)\cdot(0.05) = 1 + \frac{5}{2}\frac{\sqrt{3}}{2} = \frac{4+5\sqrt{3}}{4}$$
$$\cong 3.165\ in/s$$

b) From the specification for $A(t)$ in the relations, we have

$$\left.\frac{dA}{dt}(t)\right|_{t=t_0} = \left.\left(\frac{d}{dt}\frac{\left(x(t)h(t)\right)}{2}\right)\right|_{t=t_0}$$
$$= x'(t_0)h(t_0) + x(t_0)\frac{dh}{dt}(t_0)$$

Using the specifications from the given information, we have

$$h(t_0) = y(t_0)\sin\left(\phi(t_0)\right) = 50\sin(\pi/6) = 25\ in$$

Using this result, the specifications from the given information, and the result of part a), we obtain

$$\left.\frac{dA}{dt}(t)\right|_{t=t_0} = x'(t_0)h(t_0) + x(t_0)\frac{dh}{dt}(t_0)$$
$$= 3\cdot 25 + 40\cdot\frac{4+5\sqrt{3}}{4} = 115 + 50\sqrt{3}$$
$$\cong 201.6\ in^2/s$$

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