SM221 Exam 3, Spring 2007
Solutions

```
>
>
> restart;
> with(plots):
Warning, the name changecoords has been redefined
>
```

1. A half ballball of diameter 4 sits centered at the origin. A conical wedge

z = sqrt(3(x^2 + y^2))

is drilled out of it, as shown in Figure 1.

a. Set up an iteration of integrals that will produce the volume of the solid.

b. What is the volume of the solid?

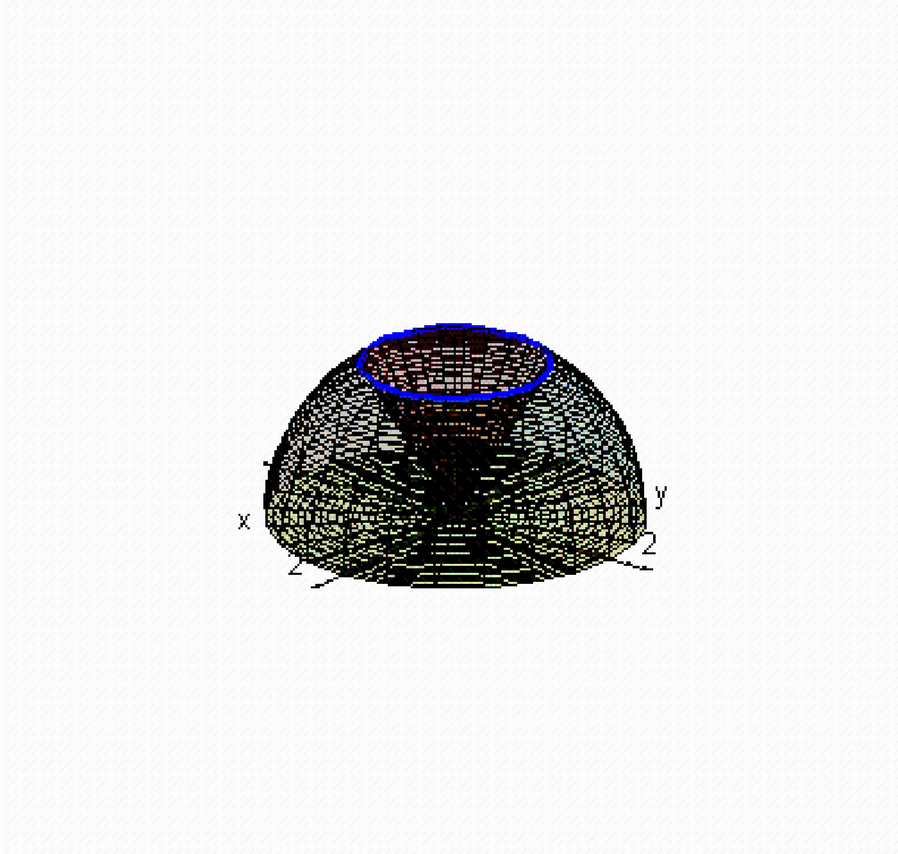
c. What is the iteration of integrals that will produce the area of the spherical surface.

```
>
```

Figure 1

```
>
Surf11:=plot3d([2,theta,phi],theta=0..2*Pi,phi=arccot(sqrt(
3))..Pi/2,coords=spherical,transparency=0.8):
>
Surf12:=plot3d([rho,theta,arccot(sqrt(3))],theta=0..2*Pi,rh
o=0..2,coords=spherical,color=red,transparency=0.8):
>
Surf13:=plot3d([r,theta,0],r=0..2,theta=0..2*Pi,coords=cyli
ndrical,transparency=0.8,color=yellow):
>
Curve14:=spacecurve([1,theta,sqrt(3)],theta=0..2*Pi,coords=
cylindrical,color=blue,thickness=2):
>
display(Surf11,Surf12,Surf13,Curve14,scaling=constrained,ax
es=normal,view=[-2.5..2.5,-
2.5..2.5,0..2],labels=["x","y","z"],lightmodel=light1,title
="Figure 1");
```

Figure 1



Solutions

a. 1/2-Ball of radius 2, center at origin: bounding surface

```
>
```

$$x^2 + y^2 + z^2 = 4, z >= 0$$

Go with spherical coords:

$$rho = 2$$

Wedge is cone:

$$z = sqrt(3 r^2)/2=sqrt(3)*r$$
$$rho cos(phi) = sqrt(3) rho sin(phi)$$
$$cot(phi) = sqrt(3)$$
$$phi = arccot(sqrt(3)); phi = Pi/6$$

So integrating in spherical coords

$$dV = (rho)^2 sin(phi) d rho d phi d theta$$

The 3pl integral is

```
> Int(Int(Int(rho^2*sin(phi),rho=0..2),phi =
arccot(sqrt(3))..Pi/2),theta=0..2*Pi);
```

$$\int_0^{2\pi}\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}\int_0^2 \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

b) evaluating,

```
> Int(Int(int(rho^2*sin(phi),rho=0..2),phi =
arccot(sqrt(3))..Pi/2),theta=0..2*Pi);
```

$$\int_0^{2\pi}\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} \frac{8}{3}\sin(\phi)\, d\phi\, d\theta$$

```
> Int(int(int(rho^2*sin(phi),rho=0..2),phi =
arctan(3)..Pi/2),theta=0..2*Pi);
```

$$\int_0^{2\pi} \frac{4}{15}\sqrt{10}\, d\theta$$

```
> int(int(int(rho^2*sin(phi),rho=0..2),phi =
arccot(sqrt(3))..Pi/2),theta=0..2*Pi);
```

$$\frac{8}{3}\sqrt{3}\,\pi$$

```
>
> evalf(%);
```

$$31.26555574$$

```
>
```

For reference,

```
> evalf(arccot(sqrt(3)));
```

$$0.5235987758$$

Depict the rho-theta integration, phi fixed (phi = Pi/4)

```
> [rho,theta,Pi/3];
```

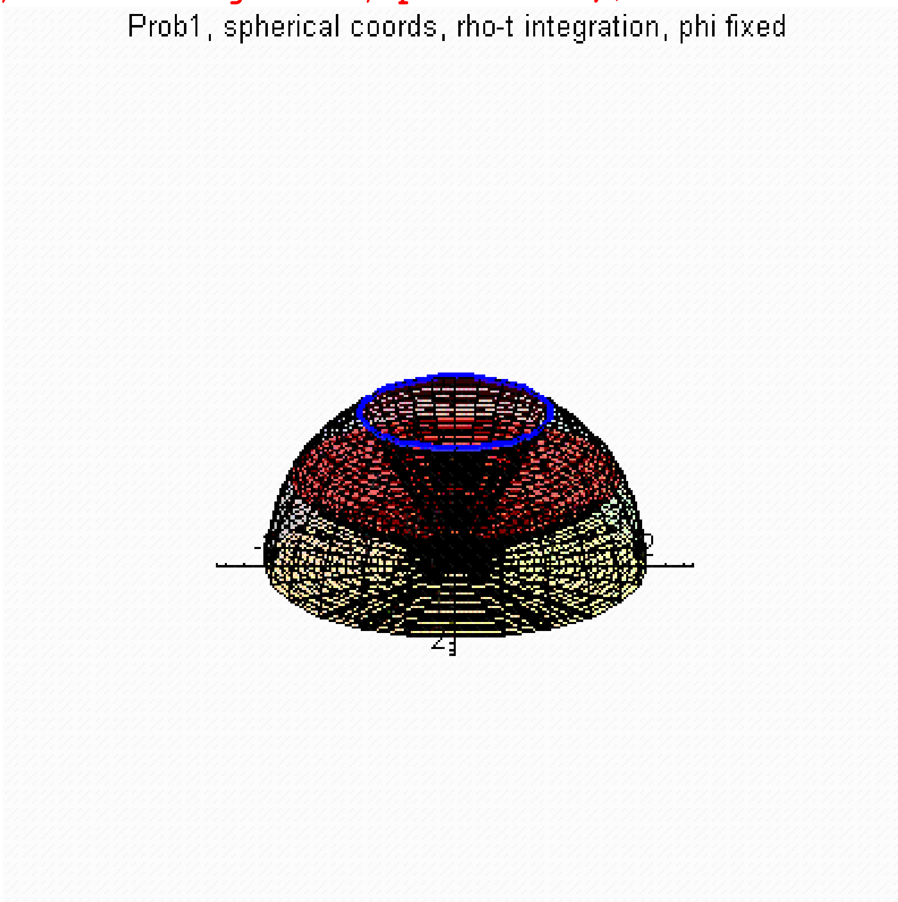
$$\left[\rho, \theta, \frac{1}{3}\pi\right]$$

```
> RConePi4:=unapply(%,(rho,theta));
```

$$RConePi4 := (rho, theta) \rightarrow \left[\rho, \theta, \frac{1}{3}\pi\right]$$

```
>
```

```
ConePi4:=plot3d(RConePi4(rho,theta),rho=0..2,theta=0..2*Pi,
color=red,coords=spherical,transparency=0.5):
>
display(Surf11,Surf12,Surf13,Curve14,ConePi4,scaling=constr
ained,axes=normal,view=[-2.5..2.5,-
2.5..2.5,0..2],labels=["x","y","z"],title="Prob1, spherical
coords, rho-t integration, phi fixed");
```



\>

Alternative Go with cylindrical coords

sphere:

r^2+z^2 = 4

Wedge is cone:

z = sqrt(3)*r

Intersection:

4*r^2 = 4; r = 1, or z = sqrt(3)

Bottom disk:

r<= 2, z = 0

So integrating in cylindrical coords

dV = r dr d theta dz

The 3pl integral is

```
> Int(Int(Int(r,r=z/sqrt(3)..sqrt(4-z^2)),theta =
0..2*Pi),z=0..sqrt(3));
```

$$\int_{0}^{\sqrt{3}} \int_{0}^{2\pi} \int_{\frac{1}{3} z \sqrt{3}}^{\sqrt{4-z^2}} r \, dr \, d\theta \, dz$$

b) evaluating,

```
> Int(Int(int(r,r=z/sqrt(3)..sqrt(4-z^2)),theta =
0..2*Pi),z=0..sqrt(3));
```

$$\int_{0}^{\sqrt{3}} \int_{0}^{2\pi} 2 - \frac{2}{3} z^2 \, d\theta \, dz$$

```
> Int(int(int(r,r=z/sqrt(3)..sqrt(4-z^2)),theta =
0..2*Pi),z=0..sqrt(3));
```

$$\int_{0}^{\sqrt{3}} 4\pi - \frac{4}{3} z^2 \pi \, dz$$

```
> int(int(int(r,r=z/sqrt(3)..sqrt(4-z^2)),theta =
0..2*Pi),z=0..sqrt(3));
```

$$\frac{8}{3} \pi \sqrt{3}$$

```
>
> evalf(%);
```

$$14.51039492$$

Depict the r-theta integration (z fixed) in the figure.

```
> [r*cos(t),r*sin(t),1];
```

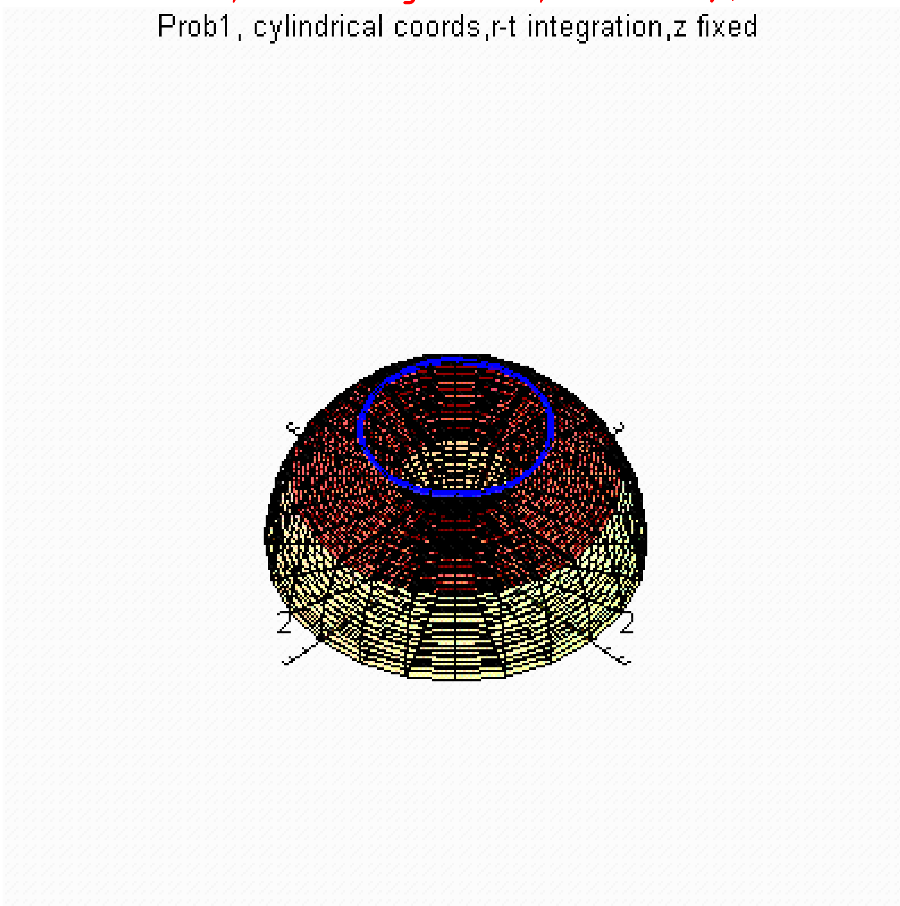
$$[r \cos(t), r \sin(t), 1]$$

```
> d1:=unapply(%,(r,t));
```

$$d1 := (r, t) \rightarrow [r \cos(t), r \sin(t), 1]$$

```
>
Disk1:=plot3d(d1(r,t),r=1/sqrt(3)..sqrt(3),t=0..2*Pi,color=
red,transparency=0.5):
>
display(Surf11,Surf12,Surf13,Curve14,Disk1,scaling=constrai
ned,axes=normal,view=[-2.5..2.5,-
2.5..2.5,0..2],labels=["x","y","z"],title="Prob1,
```

```
cylindrical coords,r-t integration,z fixed");
```



\>

2. A solid is bounded by the surfaces

$$z + y^2 = 1$$
$$x - y = 1$$
$$x = 0$$
$$y = 0$$
$$z = 0$$

Matter is distributed throughout the solid with a density directly proportional to the distance from the xy-plane,

$$f(x,y,z) = 5z \ .$$

a. Set up an iteration of integrals for the total mass of the solid.
b. Compute the mass of the solid.

Solution
a)

\>

>

From the figure, we integrate in Cartesian coords in the order dV = dz dx d6y

>

```
> Int(Int(Int(5*z,z=0..1-y^2),x=0..1+y),y=0..1);
```

$$\int_0^1 \int_0^{1+y} \int_0^{1-y^2} 5\,z\,dz\,dx\,dy$$

b) Evaluating,

```
> Int(Int(int(5*z,z=0..1-y^2),x=0..1+y),y=0..1);
```

$$\int_0^1 \int_0^{1+y} \frac{5}{2}(1-y^2)^2\,dx\,dy$$

```
> Int(int(int(5*z,z=0..1-y^2),x=0..1+y),y=0..1);
```

$$\int_0^1 \frac{5}{2}(1-y^2)^2(1+y)\,dy$$

```
> int(int(int(5*z,z=0..1-y^2),x=0..1+y),y=0..1);
```

$$\frac{7}{4}$$

3. A four-bladed propeller is shown in cross section as Figure 2. Each blade is 1 meter long. In polar coordinates, the equation for the curve bounding the edge of the cross section is

r = cos(2 theta)

Assume the propeller has a uniform area mass density of 10 . The total mass of the propeller is the integral of its density over the area of all four blades.

a. Set up the multiple integral for the mass of the propeller.

b. Evaluate the integral to compute the mass.

Solutions

a) We compute the mass of one blade and multiply by four. The limits on the angle:

>

0 = cos(2 theta)
2 theta = +/- Pi/2
theta = -Pi/4; theta = Pi/4

We compute

int(10 da), da = r dr dtheta

```
>
> Int(Int(10*r,r=0..cos(2*theta)),theta=-Pi/4..Pi/4);
>
```

$$\int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} \int_{0}^{\cos(2\,\theta)} 10\, r \, dr \, d\theta$$

b) Evaluating,

```
>
> Int(int(10*r,r=0..cos(2*theta)),theta=-Pi/4..Pi/4);
```

$$\int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} 5\cos(2\,\theta)^2 \, d\theta$$

```
> int(int(10*r,r=0..cos(2*theta)),theta=-Pi/4..Pi/4);
```

$$\frac{5}{4}\pi$$

```
>
> evalf(%);
>
```

$$3.926990818$$

```
>
```

4. a) Evaluate the mid point estimation for the integral whose table is given
b) Reverse the orders of integration.

```
>
```

a) For m = 2, n = 2, the midpoints of interest are: f11 = f(2,15) = 10; f21 = f(6,15) = 20; f12 = f(2,25) = 25; f22 = f(6,25) = 5

The area decrement DA = 4*10 = 40

```
> f4est:=[10,20,25,5];
```

$$f4est := [10, 20, 25, 5]$$

```
> intEst:=sum(f4est[i],i=1..4)*40;
```

$$intEst := 2400$$

```
>
```

b) Sketch the regions and reverse the order of integrations on

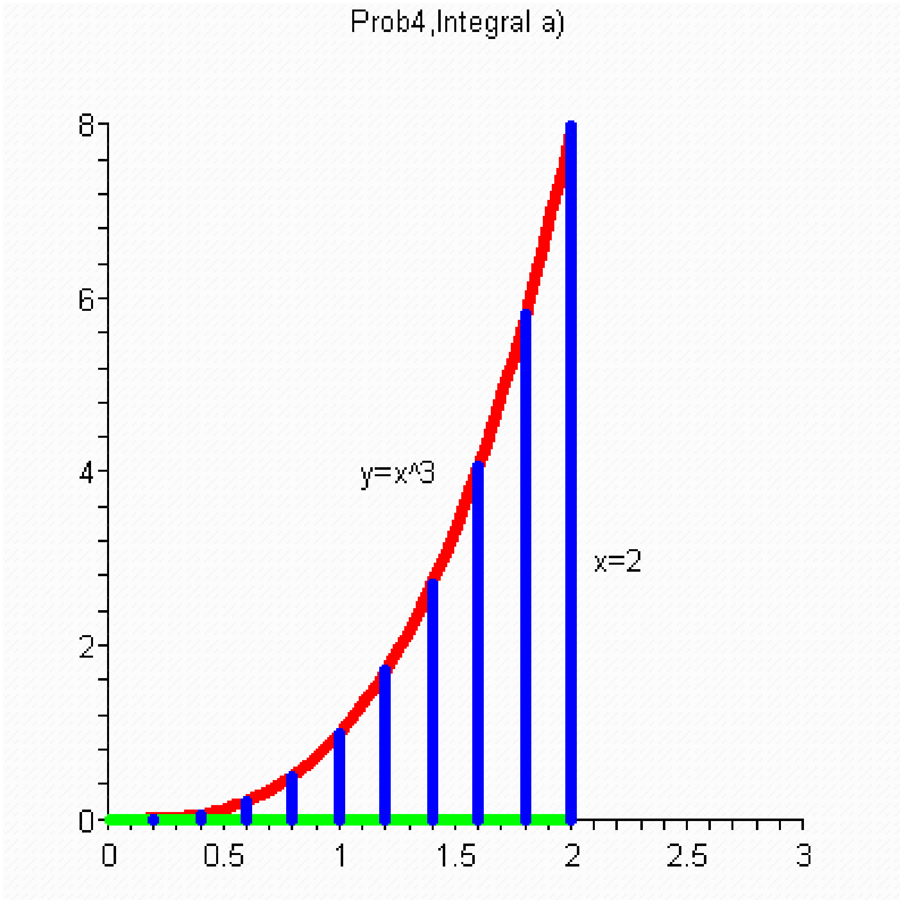
```
>
Int(Int(g(x,y),y=0..x^3),x=1..2);Int(Int(h(x,y),x=y^3+1..sqrt(y)+1),y=0..1);
```

$$\int_{1}^{2} \int_{0}^{x^3} g(x, y)\, dy\, dx$$

$$\int_0^1 \int_{y^3+1}^{\sqrt{y}+1} h(x, y)\, dx\, dy$$

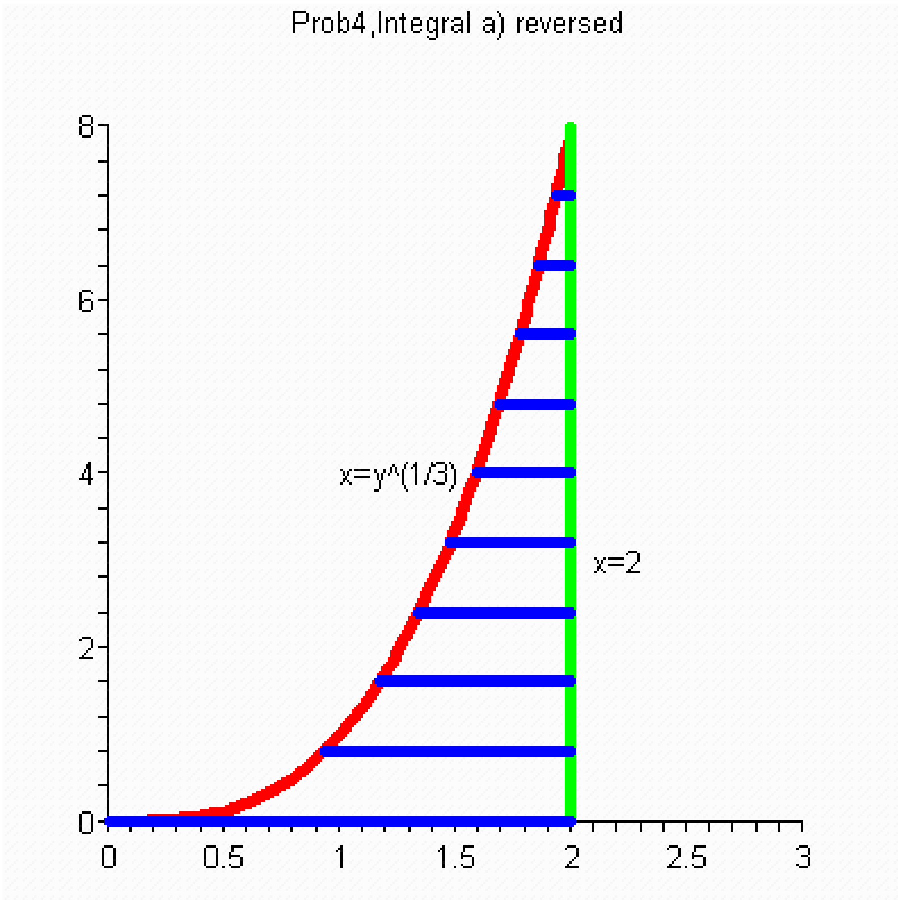
```
> bottomcurve := x -> 0:
topcurve:= x -> x^3:
leftend := 0: rightend := 2:
print(`region for `, Int(g(x,y),A)=Int(Int(g(x,y),
y=bottomcurve(x)..topcurve(x)), x=leftend..rightend));
lines := {}:
for i from 0 to 10 do
tempx := leftend + i/10*(rightend - leftend):
lines := lines union
{[[tempx,bottomcurve(tempx)], [tempx,topcurve(tempx)]]};
od:
plots[display]([plot({[x,bottomcurve(x),x=leftend..rightend
],
[x,topcurve(x),x=leftend..rightend]}),textplot([[1.25,4,"y=
x^3"],[2.2,3,"x=2"]],color=black),
plot(lines, color=blue)],
thickness=3,view=[0..3,0..8],title="Prob4,Integral a)");
```

$$\textit{region for }, \int g(x, y)\, dA = \int_0^2 \int_0^{x^3} g(x, y)\, dy\, dx$$




```
>
> leftcurve := y -> y^(1/3):
rightcurve := y -> 2:
bottomend:= 0: topend := 8:
print(`region for `,Int(g(x,y),A)= Int(Int(g(x,y),
x=leftcurve(y)..rightcurve(y)), y=bottomend..topend));
lines := {}:
for i from 0 to 10 do
tempy := bottomend + i/10*(topend - bottomend):
lines := lines union
{[[leftcurve(tempy),tempy], [rightcurve(tempy),tempy]]};
od:
display([plot({[leftcurve(y),y,y=bottomend..topend],
[rightcurve(y), y,y=bottomend..topend]}),
textplot([[1.25,4,"x=y^(1/3)"],[2.2,3,"x=2"]],color=black),
plot(lines, color=blue)],
thickness=3,view=[0..3,0..8],title="Prob4,Integral a)
reversed");
```

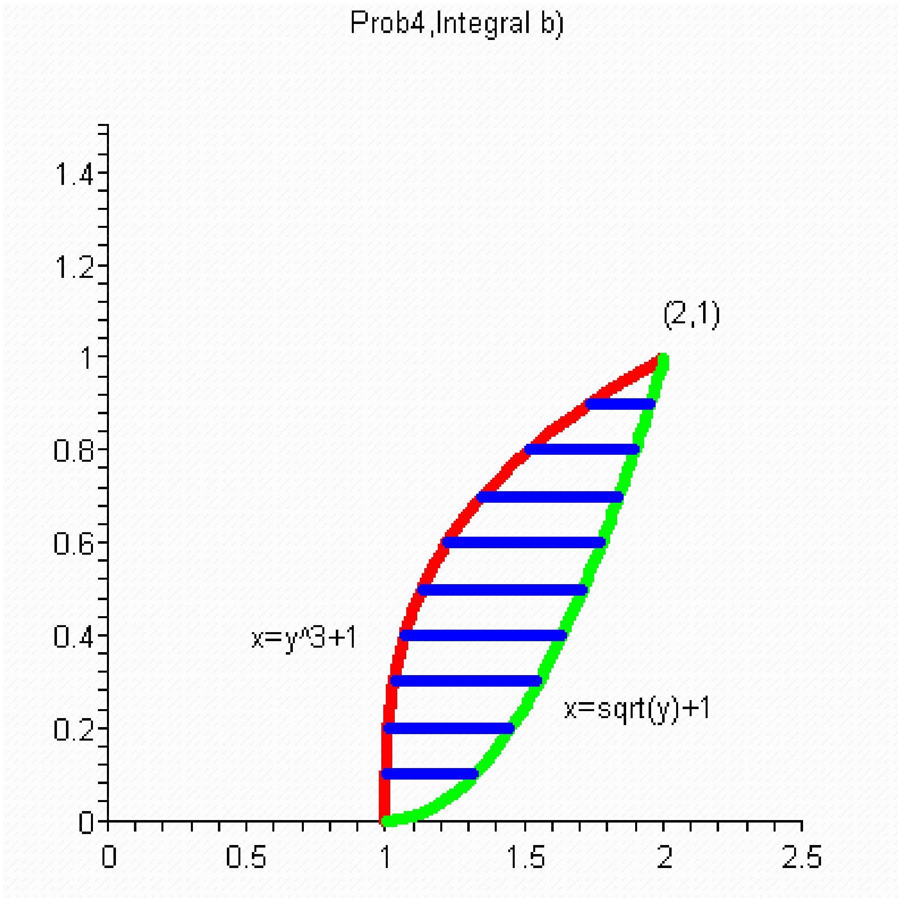
$$\text{region for}\ , \int g(x,y)\, dA = \int_0^8 \int_{y^{(1/3)}}^2 g(x,y)\, dx\, dy$$




```
>
> leftcurve := y -> y^3+1:
rightcurve := y -> sqrt(y)+1:
bottomend:= 0: topend := 1:
print(`region for `,Int(h(x,y),A)= Int(Int(h(x,y),
x=leftcurve(y)..rightcurve(y)), y=bottomend..topend));
lines := {}:
for i from 0 to 10 do
tempy := bottomend + i/10*(topend - bottomend):
lines := lines union
{[[leftcurve(tempy),tempy], [rightcurve(tempy),tempy]]};
od:
display([plot({[leftcurve(y),y,y=bottomend..topend],
[rightcurve(y), y,y=bottomend..topend]}),
textplot([[0.7,0.4,"x=y^3+1"],[1.9,0.25,"x=sqrt(y)+1"],[2.1
,1.1,"(2,1)"]],color=black),
plot(lines, color=blue)],
thickness=3,view=[0..2.5,0..1.5],title="Prob4,Integral
```

```
b)");
```

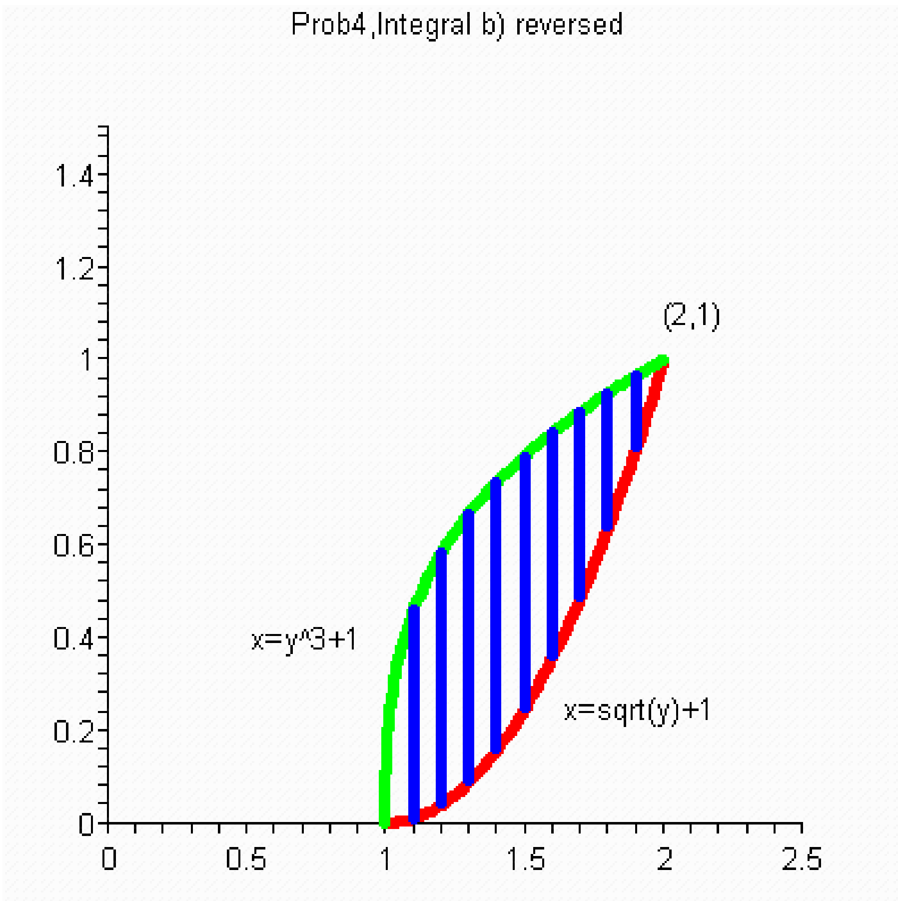
$$region\ for\ ,\ \int h(x,y)\,dA = \int_0^1 \int_{y^3+1}^{\sqrt{y}+1} h(x,y)\,dx\,dy$$




```
>
>
> bottomcurve := x -> (x-1)^2:
topcurve:= x -> (x-1)^(1/3):
leftend := 1: rightend := 2:
print(`region for `, Int(h(x,y),A)=Int(Int(h(x,y),
y=bottomcurve(x)..topcurve(x)), x=leftend..rightend));
lines := {}:
for i from 0 to 10 do
tempx := leftend + i/10*(rightend - leftend):
lines := lines union
{[[tempx,bottomcurve(tempx)], [tempx,topcurve(tempx)]]};
od:
plots[display]([plot({[x,bottomcurve(x),x=leftend..rightend
],
```

```
[x,topcurve(x),x=leftend..rightend]}),textplot([[0.7,0.4,"x
=y^3+1"],[1.9,0.25,"x=sqrt(y)+1"],[2.1,1.1,"(2,1)"]],color=
black),
plot(lines, color=blue)],
thickness=3,view=[0..2.5,0..1.5],title="Prob4,Integral b)
reversed");
```

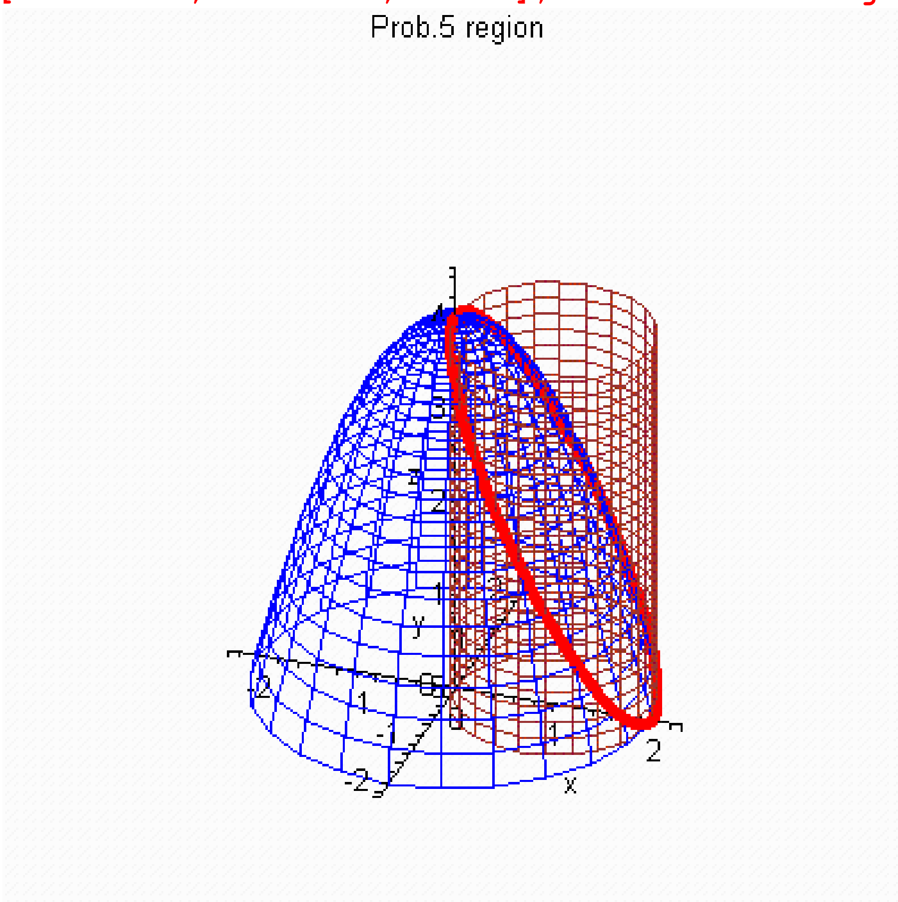
$$\text{region for}\,,\ \int h(x,y)\,dA = \int_{1}^{2}\int_{(x-1)^2}^{(x-1)^{(1/3)}} h(x,y)\,dy\,dx$$



>
>
5. Sketch the region and set up the iteration of integrals that produces the area of the surface x^2 + y^2 + z = 4 that is contained within the surface (x-1)^2 + y^2 = 1.

Solution
The surfaces are: a circular paraboloid, upside down, vertex (0,0,4); an offset cylinder, axis (1,0,z), radius 1. The curve of intersection of the two surfaces is the curve (x-1)^2 + y^2 =1, or x^2 + y^2 = 2x, and z = 4 - 2*x. We display this extent in cylindrical coordinates.

```
>
>
Surf1:=plot3d([1+cos(t),sin(t),z],t=0..2*Pi,z=0..4,style=wireframe,transparency=0.6,color=brown):
> Surf2:=plot3d([r*cos(t),r*sin(t),4-r^2],r=0..2,t=0..2*Pi,style=wireframe,color=blue):
> Curve3:=spacecurve([1+cos(t),sin(t),2-2*cos(t)],t=0..2*Pi,thickness=3,color=red):
>
display(Surf1,Surf2,Curve3,axes=normal,labels=["x","y","z"],view=[-2.3..2.3,-2.3..2.3,0..4.5],title="Prob.5 region");
```



We create a motion over the surface segment, display the line elements, compute the surface area element, and display the region of integration.

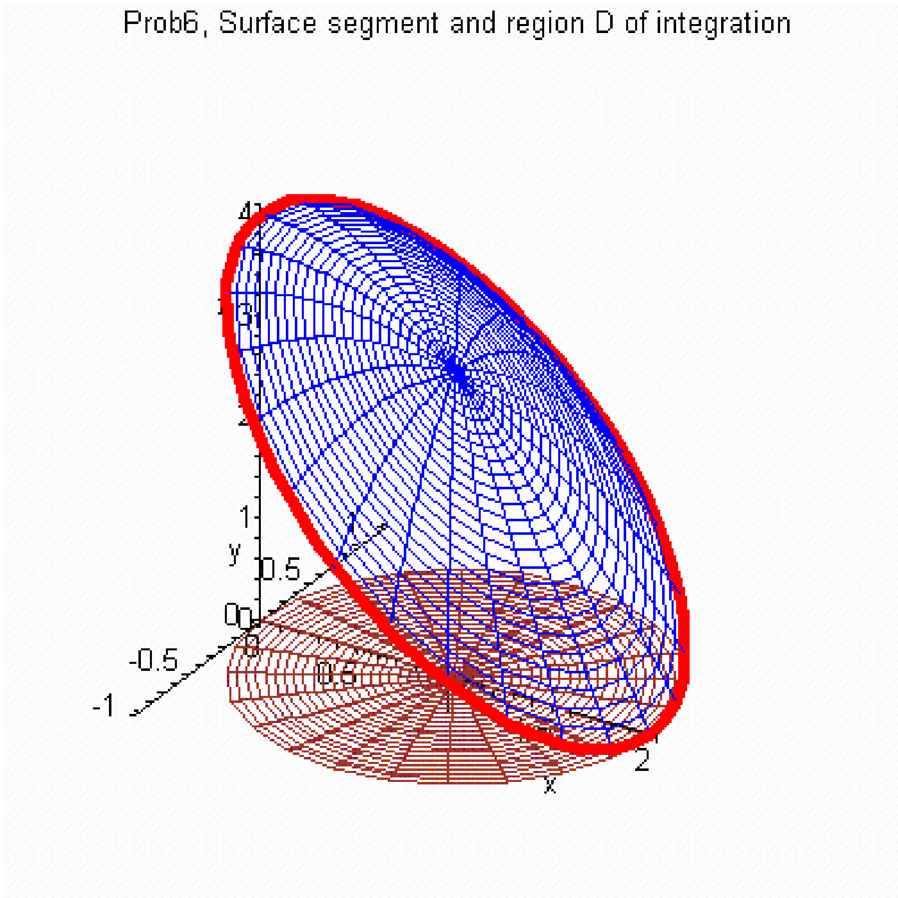
```
>
> f6:=(x,y)->4-(x^2+y^2);
```

$$f6 := (x, y) \to 4 - x^2 - y^2$$

Create the parameterized surface over the base:

```
>
Surf61:=plot3d([1+r*cos(t),r*sin(t),f6(1+r*cos(t),r*sin(t))],t=0..2*Pi,r=0..1,style=wireframe,color=blue):
>
```

```
Curve62:=spacecurve([1+cos(t),sin(t),f6(1+cos(t),sin(t))],t
=0..2*Pi,color=red,thickness=3):
>
Base63:=plot3d([1+r*cos(t),r*sin(t),0],t=0..2*Pi,r=0..1,sty
le=wireframe,transparency=0.6,color=brown):
>
display(Surf61,Curve62,Base63,axes=normal,labels=["x","y","
z"],title="Prob6, Surface segment and region D of
integration");
```



>

Move on the surface: (x,y,4-x^2-y^2)

```
> with(plottools):
Warning, the assigned name arrow now has a global binding
> [x,y,f6(x,y)];
```

$$[x, y, 4 - x^2 - y^2]$$

```
> r6:=unapply(%,(x,y));
```

$$r6 := (x, y) \to [x, y, 4 - x^2 - y^2]$$

```
> diff(r6(x,y),x);
```

$$[1, 0, -2\,x]$$

```
> v61:=unapply(%,(x,y));
```

$$v61 := (x, y) \to [1, 0, -2\,x]$$

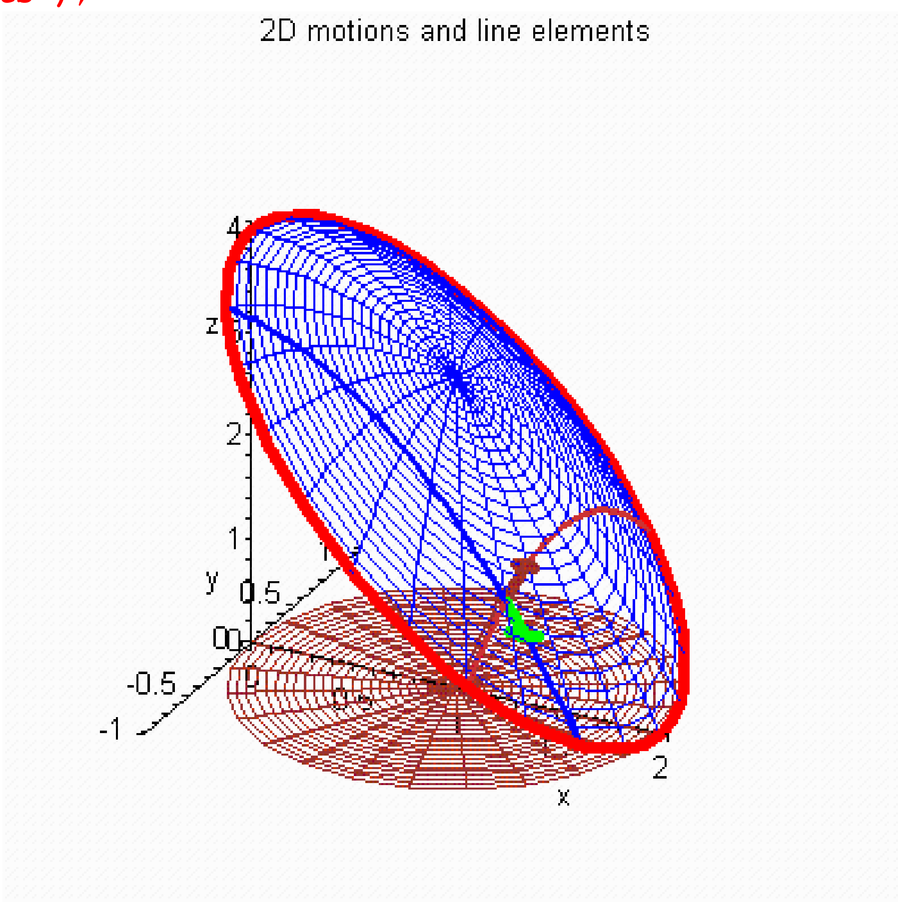
```
> diff(r6(x,y),y);
```

$$[0, 1, -2\,y]$$

```
> v62:=unapply(%,(x,y));
```

$$v62 := (x, y) \to [0, 1, -2\,y]$$

```
> SC61:=spacecurve(r6(x,-0.5),x=1-sqrt(1-0.25)..1+sqrt(1-
0.25),thickness=2,color=blue):
> SC62:=spacecurve(r6(1.5,y),y=-sqrt(3-2.25)..sqrt(3-
2.25),thickness=2,color=orange):
> SV61:=arrow(r6(1.5,-0.5),vector(0.1*v61(1.5,-
0.5)),0.2,0.2,0.1,arrow,color=green,thickness=3):
> SV62:=arrow(r6(1.5,-0.5),vector(0.2*v62(1.5,-
0.5)),0.2,0.2,0.1,arrow,color=brown,thickness=3):
>
>
display(Surf61,Curve62,Base63,SC61,SC62,SV61,SV62,axes=norm
al,labels=["x","y","z"],title="2D motions and line
elements");
```



```
>
```

The area element: v1(x,y) = [1,0,-2x], v2(x,y) = [0,1,-2y], crossProd(v1(x,y),v2(x,y)) = [2x,2y,1], dS = sqrt(1 + 4(x^2+y^2)) dydx

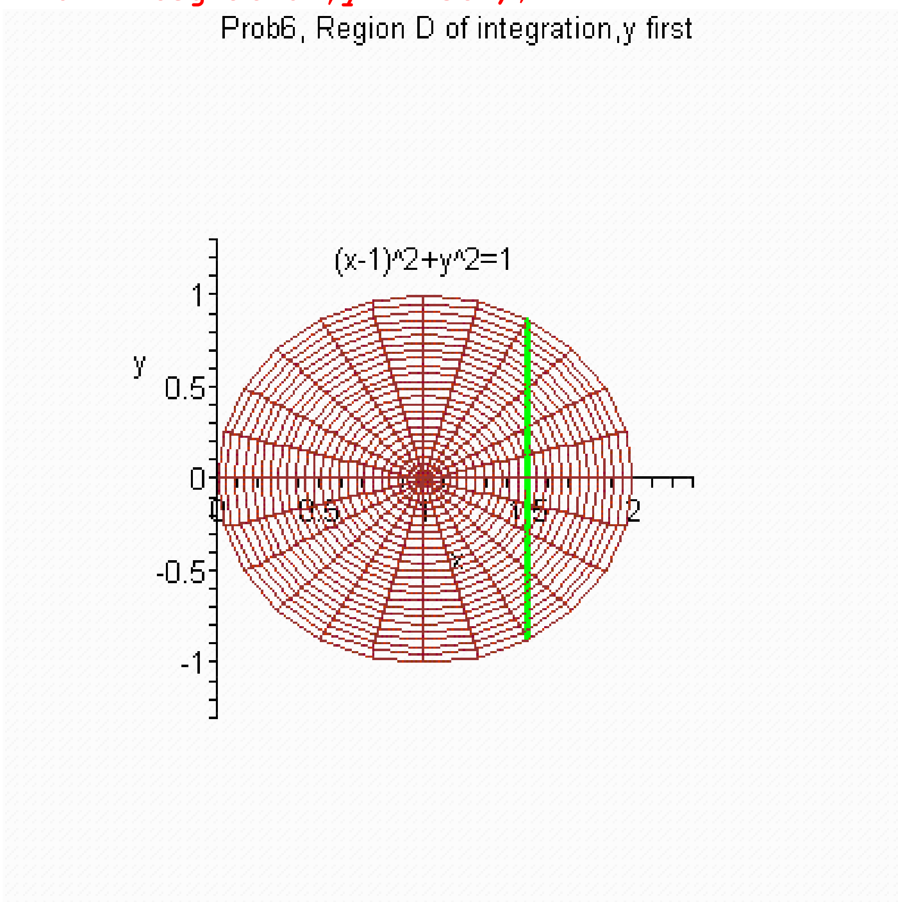
```
> Int(sqrt(1+4*(x^2+y^2)),A);
```

$$\int \sqrt{1+4x^2+4y^2}\, dA$$

```
>
>
```

over region:

```
> Curve64:=spacecurve([1.5,y,0],y=-sqrt(3-2.25)..sqrt(3-2.25),thickness=2,color=green):
> TextPlot61:=textplot3d([1,1.2,0,`(x-1)^2+y^2=1`],color=black):
>
display(Base63,Curve64,TextPlot61,axes=normal,labels=["x","y","z"],view=[0..2.3,-1.3..1.3,-0.1..0.1],title="Prob6, Region D of integration,y first");
```




```
>
```

In Cartesian coordinates, our integral becomes apparent. solve for y: y = +/-sqrt(1-(x-1)^2) = +/-sqrt(2*x - x^2)

```
> Int(Int(sqrt(1+4*(x^2+y^2)),y=-sqrt(1-(x-1)^2)..sqrt(1-(x-1)^2)),x=0..2);
```

$$\int_0^2 \int_{-\sqrt{-x^2+2x}}^{\sqrt{-x^2+2x}} \sqrt{1+4x^2+4y^2}\, dy\, dx$$

For fun, let's evaluate the integral:

```
> Int(int(sqrt(1+4*(x^2+y^2)),y=-sqrt(1-(x-1)^2)..sqrt(1-(x-1)^2)),x=0..2);
```

$$\int_0^2 \sqrt{-x(x-2)}\,\sqrt{1+8x} - \frac{1}{4}\ln(-2\sqrt{-x(x-2)}+\sqrt{1+8x}) - \ln(-2\sqrt{-x(x-2)}+\sqrt{1+8x})\,x^2$$
$$+ \frac{1}{4}\ln(2\sqrt{-x(x-2)}+\sqrt{1+8x}) + \ln(2\sqrt{-x(x-2)}+\sqrt{1+8x})\,x^2\, dx$$

Ghastly, but a quadrature

```
> int(int(sqrt(1+4*(x^2+y^2)),y=-sqrt(1-(x-1)^2)..sqrt(1-(x-1)^2)),x=0..2);
```

$$\int_0^2 \sqrt{-x(x-2)}\,\sqrt{1+8x} - \frac{1}{4}\ln(-2\sqrt{-x(x-2)}+\sqrt{1+8x}) - \ln(-2\sqrt{-x(x-2)}+\sqrt{1+8x})\,x^2$$
$$+ \frac{1}{4}\ln(2\sqrt{-x(x-2)}+\sqrt{1+8x}) + \ln(2\sqrt{-x(x-2)}+\sqrt{1+8x})\,x^2\, dx$$

Does not yield as an exact quadrature. We compute it numerically

```
> evalf(int(int(sqrt(1+4*(x^2+y^2)),y=-sqrt(1-(x-1)^2)..sqrt(1-(x-1)^2)),x=0..2));
```

$$7.904176661$$

FIN!!

NOT USED

4. A surface is described by the parametric equation

x(u,v) = u + v
y(u,v) = u^2 + v^2
z(u,v) = 2 u v

and shown in Figure 3. Over the extent 0 <= u <= 1, 0 <= v <=1, set up an iteration of integrals for the surface area. Do not evaluate the iteration of integrals.

Solution

We build the area element, dS = |N|dudv

```
> with(linalg):
Warning, the protected names norm and trace have been redefined and
unprotected

> r:=(u,v)->[u+v,u^2+v^2,2*u*v];
```

$$r := (u, v) \to [u + v, u^2 + v^2, 2\,u\,v]$$

```
>
> diff(r(u,v),u);
```

$$[1, 2\,u, 2\,v]$$

```
> Dru:=unapply(%,u,v);
```

$$Dru := (u, v) \to [1, 2\,u, 2\,v]$$

```
>
> diff(r(u,v),v);
```

$$[1, 2\,v, 2\,u]$$

```
> Drv:=unapply(%,u,v);
```

$$Drv := (u, v) \to [1, 2\,v, 2\,u]$$

```
>
> crossprod(Dru(u,v),Drv(u,v));
```

$$[4\,u^2 - 4\,v^2, 2\,v - 2\,u, 2\,v - 2\,u]$$

```
>
> N:=(u,v)->[4*(u^2-v^2),2*(v-u),2*(v-u)];
```

$$N := (u, v) \to [4\,u^2 - 4\,v^2, 2\,v - 2\,u, 2\,v - 2\,u]$$

```
>
> assume(u>=0): assume(v>=0):
>
> dotprod(N(u,v),N(u,v));
```

$$(4\,u\sim^2 - 4\,v\sim^2)^2 + 2\,(2\,v\sim - 2\,u\sim)^2$$

```
> sqrt(%);
```

$$2\sqrt{4\,u\sim^4 - 8\,u\sim^2\,v\sim^2 + 4\,v\sim^4 + 2\,v\sim^2 - 4\,v\sim\,u\sim + 2\,u\sim^2}$$

```
> magN:=unapply(%,u,v);
```

$$magN := (u\sim, v\sim) \rightarrow 2\sqrt{4u\sim^4 - 8u\sim^2 v\sim^2 + 4v\sim^4 + 2v\sim^2 - 4u\sim v\sim + 2u\sim^2}$$

\>

We integrage dS = magN(u,v)dudv over the extent:

\>

```
> Int(Int(magN(u,v),u=0..1),v=0..1);
```

$$\int_0^1 \int_0^1 2\sqrt{4u\sim^4 - 8u\sim^2 v\sim^2 + 4v\sim^4 + 2v\sim^2 - 4v\sim u\sim + 2u\sim^2}\, du\sim\, dv\sim$$

\>

for grins, evaluate it

\>

```
> Int(int(magN(u,v),u=0..1),v=0..1);
```

$$\int_0^1 \frac{1}{3(-1+v\sim)\sqrt{8v\sim + 4v\sim^2 + 6}\, v\sim \sqrt{2+4v\sim^2}} \Big(2\sqrt{6 - 4v\sim - 6v\sim^2 + 4v\sim^4}\, v\sim^2 \sqrt{2+4v\sim^2}$$

$$\sqrt{8v\sim + 4v\sim^2 + 6} + 4\sqrt{6 - 4v\sim - 6v\sim^2 + 4v\sim^4}\, v\sim^3 \sqrt{2+4v\sim^2}\sqrt{8v\sim + 4v\sim^2 + 6}$$

$$- 3\sqrt{6 - 4v\sim - 6v\sim^2 + 4v\sim^4}\, v\sim \sqrt{2+4v\sim^2}\sqrt{8v\sim + 4v\sim^2 + 6}$$

$$+ 6\sqrt{6 - 4v\sim - 6v\sim^2 + 4v\sim^4}\, v\sim^2 \sqrt{2+4v\sim^2}\ln(\sqrt{2} + \sqrt{2}\, v\sim + \sqrt{4v\sim + 2v\sim^2 + 3})$$

$$- \sqrt{4v\sim^4 + 2v\sim^2}\sqrt{8v\sim + 4v\sim^2 + 6}\sqrt{2+4v\sim^2}$$

$$+ \sqrt{4v\sim^4 + 2v\sim^2}\sqrt{8v\sim + 4v\sim^2 + 6}\, v\sim \sqrt{2+4v\sim^2}$$

$$+ 4\sqrt{4v\sim^4 + 2v\sim^2}\sqrt{8v\sim + 4v\sim^2 + 6}\sqrt{2+4v\sim^2}\, v\sim^2$$

$$- 4\sqrt{4v\sim^4 + 2v\sim^2}\sqrt{8v\sim + 4v\sim^2 + 6}\sqrt{2+4v\sim^2}\, v\sim^3$$

$$+ 6\sqrt{4v\sim^4 + 2v\sim^2}\sqrt{8v\sim + 4v\sim^2 + 6}\, v\sim \ln(\sqrt{2}\, v\sim + \sqrt{1+2v\sim^2})$$

$$- 6\sqrt{4v\sim^4 + 2v\sim^2}\sqrt{8v\sim + 4v\sim^2 + 6}\, v\sim^2 \ln(\sqrt{2}\, v\sim + \sqrt{1+2v\sim^2}) - 3 \Big($$

$$\left\{\begin{array}{cc} 0 & v\sim \leq 0 \\ \frac{8}{3}\sqrt{2}\sqrt{8v\sim^2+1}\; v\sim^2 - \frac{2}{3}\sqrt{2}\sqrt{8v\sim^2+1} + 4v\sim \ln(2\sqrt{2}\; v\sim + \sqrt{8v\sim^2+1}) & v\sim < 1 \\ 0 & 1 \leq v\sim \end{array}\right)$$

$$\sqrt{8v\sim + 4v\sim^2 + 6}\, v\sim \sqrt{2+4v\sim^2} + 3 \Bigg($$

$$\left\{\begin{array}{cc} 0 & v\sim \leq 0 \\ \frac{8}{3}\sqrt{2}\sqrt{8v\sim^2+1}\; v\sim^2 - \frac{2}{3}\sqrt{2}\sqrt{8v\sim^2+1} + 4v\sim \ln(2\sqrt{2}\; v\sim + \sqrt{8v\sim^2+1}) & v\sim < 1 \\ 0 & 1 \leq v\sim \end{array}\right)$$

$$\sqrt{8v\sim + 4v\sim^2 + 6}\, v\sim^2 \sqrt{2+4v\sim^2}\Bigg)\, dv\sim$$

```
> int(int(magN(u,v),u=0..1),v=0..1);
```

$$\int_0^1 \frac{1}{3(-1+v\sim)\sqrt{8v\sim+4v\sim^2+6}\; v\sim \sqrt{2+4v\sim^2}}\Bigg(2\sqrt{6-4v\sim-6v\sim^2+4v\sim^4}\; v\sim^2 \sqrt{2+4v\sim^2}$$

$$\sqrt{8v\sim+4v\sim^2+6} + 4\sqrt{6-4v\sim-6v\sim^2+4v\sim^4}\; v\sim^3 \sqrt{2+4v\sim^2}\sqrt{8v\sim+4v\sim^2+6}$$

$$-3\sqrt{6-4v\sim-6v\sim^2+4v\sim^4}\; v\sim \sqrt{2+4v\sim^2}\sqrt{8v\sim+4v\sim^2+6}$$

$$+6\sqrt{6-4v\sim-6v\sim^2+4v\sim^4}\; v\sim^2 \sqrt{2+4v\sim^2}\ln(\sqrt{2}+\sqrt{2}\; v\sim + \sqrt{4v\sim+2v\sim^2+3})$$

$$-\sqrt{4v\sim^4+2v\sim^2}\sqrt{8v\sim+4v\sim^2+6}\sqrt{2+4v\sim^2}$$

$$+\sqrt{4v\sim^4+2v\sim^2}\sqrt{8v\sim+4v\sim^2+6}\; v\sim \sqrt{2+4v\sim^2}$$

$$+4\sqrt{4v\sim^4+2v\sim^2}\sqrt{8v\sim+4v\sim^2+6}\sqrt{2+4v\sim^2}\; v\sim^2$$

$$-4\sqrt{4v\sim^4+2v\sim^2}\sqrt{8v\sim+4v\sim^2+6}\sqrt{2+4v\sim^2}\; v\sim^3$$

$$+6\sqrt{4v\sim^4+2v\sim^2}\sqrt{8v\sim+4v\sim^2+6}\; v\sim \ln(\sqrt{2}\; v\sim + \sqrt{1+2v\sim^2})$$

$$-6\sqrt{4\,v\sim^4+2\,v\sim^2}\,\sqrt{8\,v\sim+4\,v\sim^2+6}\,v\sim^2\ln\left(\sqrt{2}\,v\sim+\sqrt{1+2\,v\sim^2}\right)-3\Bigg($$

$$\begin{cases} 0 & v\sim\le 0 \\ \frac{8}{3}\sqrt{2}\,\sqrt{8\,v\sim^2+1}\,v\sim^2-\frac{2}{3}\sqrt{2}\,\sqrt{8\,v\sim^2+1}+4\,v\sim\ln\left(2\sqrt{2}\,v\sim+\sqrt{8\,v\sim^2+1}\right) & v\sim<1 \\ 0 & 1\le v\sim \end{cases}\Bigg)$$

$$\sqrt{8\,v\sim+4\,v\sim^2+6}\,v\sim\sqrt{2+4\,v\sim^2}+3\Bigg($$

$$\begin{cases} 0 & v\sim\le 0 \\ \frac{8}{3}\sqrt{2}\,\sqrt{8\,v\sim^2+1}\,v\sim^2-\frac{2}{3}\sqrt{2}\,\sqrt{8\,v\sim^2+1}+4\,v\sim\ln\left(2\sqrt{2}\,v\sim+\sqrt{8\,v\sim^2+1}\right) & v\sim<1 \\ 0 & 1\le v\sim \end{cases}\Bigg)$$

$$\sqrt{8\,v\sim+4\,v\sim^2+6}\,v\sim^2\sqrt{2+4\,v\sim^2}\Bigg)\,dv\sim$$

>

OK, numerically evaluate the last integral:

```
> evalf(%);
```

$$1.653221066$$

>

5. Set up the area integral of 3xy over the region R bounded by the curves

(x-1)^2 + y^2 =1
(x-1) - y^2 = 0

Do not evaluate the integral.

Solution

>

GRAPHIC

>

The curves signal that we can integrate in Cartesian coordinates, integrating with respect

to x first.
The left/right bounding curves are the parabola and the right semi-circle, respectively,

x = 1 + y^2
x = 1 + sqrt(1 - y^2)

The "outer" integral is the integration with respect to y. The "point-to-point" lower/upper limits on the y integration result from intersecting the two above curves, geometrically. Analytically, this intersection is equivalent to the simultaneous solution of the equation for (x,y). As the y coordinate is pertinent to us, we combine the two equations by eliminating the x coordinate, then solve for the values of the y coordinate.

1+sqrt(1 - y^2) = 1 + y^2
1 - y^2 = y^4
y^4 + y^2 - 1= 0
y^2 = (-1 + sqrt(1 + 4))/2

(here we've excluded the "negative root" because y^2 >= 0.)

y1 = - sqrt((-1 +sqrt(5))/2)
y2 = sqrt((-1+sqrt(5))/2)

```
>
> Int(Int(3*x*y,x =(1+y^2)..(1+sqrt(1-y^2))),y=-sqrt((-1+sqrt(5))/2)..sqrt((-1+sqrt(5))/2));
```

$$\int_{-\frac{1}{2}\sqrt{-2+2\sqrt{5}}}^{\frac{1}{2}\sqrt{-2+2\sqrt{5}}} \int_{1+y^2}^{1+\sqrt{1-y^2}} 3\,x\,y\,dx\,dy$$

```
>
```

For fun, let's evaluate that integral:

```
>
> Int(int(3*x*y,x =(1+y^2)..(1+sqrt(1-y^2))),y=-sqrt((-1+sqrt(5))/2)..sqrt((-1+sqrt(5))/2));
```

$$\int_{-\frac{1}{2}\sqrt{-2+2\sqrt{5}}}^{\frac{1}{2}\sqrt{-2+2\sqrt{5}}} \frac{3}{2}\,y\left((1+\sqrt{1-y^2})^2-(1+y^2)^2\right)dy$$

That wasn't bad.

```
>
> int(int(3*x*y,x =(1+y^2)..(1+sqrt(1-y^2))),y=-sqrt((-1+sqrt(5))/2)..sqrt((-1+sqrt(5))/2));
```

$$0$$

```
>
```

Shocking, but true. Look at the graphic, and recognize that the integrand is an "odd" function of its arguments.

```
>
>
>
```