SM221 Examination 4
Spring Semester 2007 Solutions

Conditions. You may use one page of notes and a Voyage200, TI92, or any other symbolic manipulator or device to assist you in your computations. Show your development, as it will be the basis for partial credit. You may not consult or discuss the examination with any person other than the instructor.

1. A wire in the shape of a the curve $C : y = x^3$ extends from the point $(x, y) = (-1, -1)$ to the point $(x, y) = (1, 1)$. It is charged with a "linear charge density" of $f(x, y) = x + y^2$ (coulombs/meter). Set up a parameterized integral that determines the net charge on the wire. ***Do not evaluate*** the integral, just set it up.

Solutions

We seek to determine the value of the line integral of the charge density function over the curve C,

$$\int_C f(x, y)\, ds$$

by building a parameterized integral whose value is this value of interest. We use a multi-step process to erect the parameterized integral.

1) Move on the curve:

We parameterize by graph. A parameterization with extent that moves over the curve in the desired direction between the two points is

$$x(t) = t, \quad y(t) = t^3, \quad -1 \le t \le 1$$

The position vector function for the motion associated with this parameterization is

$$\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j}, \ -1 \le t \le 1$$

2) Build the scalar line element (or the decrement of arc length)

The velocity associated with this motion is

$$\mathbf{v}(t) = \frac{d}{dt}\left(\mathbf{r}(t)\right) = 1\mathbf{i} + 3t^2\mathbf{j}, \ -1 \le t \le 1$$

The vector line element is

$$d\mathbf{r}(t) = \mathbf{v}(t)\,dt = \left(1\mathbf{i} + 3t^2\mathbf{j}\right)dt, \ -1 \le t \le 1$$

The scalar line element (decrement of arc length) is the magnitude of the vector line element. Recognizing the magnitude of the velocity vector function (e.g. the speed function) is

$$v(t) = \left\|\mathbf{v}(t)\right\| = \sqrt{1^2 + \left(3t^2\right)^2} = \sqrt{1 + 9t^4}\ ,$$

the scalar line element for this motion is

$$ds(t) = \|d\mathbf{r}(t)\| = \|\mathbf{v}(t)\|\,dt = v(t)\,dt = \sqrt{1+9t^4}\,dt$$

3) Compute the charge density encountered at each instant in the motion, and the decrement of charge over a decrement of arc.

Because the linear charge density along the wire is given by location by

$$f(x,y) = x + y^2,$$

at instant t, the charge density $g(t)$ encountered on the wire by the motion is

$$g(t) = f\left(x(t), y(t)\right) = f\left(t, t^3\right) = t + \left(t^3\right)^2 = t + t^6$$

Consequently, *in terms of this motion*, over a decrement of arc ds, the charge encountered is

$$f(x,y)\,ds = \left(t + t^6\right)\sqrt{1+9t^4}\,dt$$

4) Integrate over the extent of the wire

For this motion, "To integrate over the extent of the wire" means, "To integrate over the interval of time, $-1 \le t \le 1$". Consequently, a parameterized integral that will determine the net charge on the wire is

$$\boxed{\int_C f(x,y)\,ds = \int_{t=-1}^{1} \left(t + t^6\right)\sqrt{1+9t^4}\,dt}$$

■

2. A proton is given energy by subjecting it to an electric field ***E****(x,y)*. The force exerted by the field on the proton is given by the relation

$$\boldsymbol{F}(x,y) = q\,\boldsymbol{E}(x,y),$$

where q is the charge on the proton. For this problem take $q = 1$ (esu).
Let the electric field be

$$\boldsymbol{E}(x,y) = 2 \cdot 10^4 e^{2y}\left(x\,\boldsymbol{i} + x^2\,\boldsymbol{j}\right) \text{ (V/m)}$$

In a cyclotron, by an ingenious use of magnetic fields the proton can be made to move in this electric field in the *x-y* plane along an Archimedean spiral C, described by the equations

$$y = x\tan\left(\sqrt{x^2 + y^2}\right),$$

or, in polar coordinates,

$$r = \theta,$$

and as shown in the figure 'Question 2'. The energy W gained by the proton in this process is determined by the line integral of the force exerted by the electric field on it as it moves over the curve,

$$W = \int_C \mathbf{F} \cdot \mathbf{d}r \ \text{(eV)}$$

Find the energy (eV) the proton gains as it moves from the entry port at the origin to the exit port at $(x, y) = (12\pi, 0)$.

Solution

We seek the value of the line integral of the vector field

$$\mathbf{F}(x, y) = q\mathbf{E}(x, y) = 2 \cdot 10^4 \left(xe^{2y}\mathbf{i} + x^2e^{2y}\mathbf{j} \right)$$

over the spiral curve *C* from the origin to the port. Before we set the integral up, we ask, "Might this vector field be conservative?" We apply the test: "Is the curl of the vector field everywhere zero?" For a 2D vector field, the test amount to computing

$$\frac{\partial Q(x, y)}{\partial x} - \frac{\partial P(x, y)}{\partial y}$$

for $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$. In this case, we determine that

$$\begin{aligned} \frac{\partial Q(x, y)}{\partial x} - \frac{\partial P(x, y)}{\partial y} &= 2 \cdot 10^4 \left(\frac{\partial \left(x^2e^{2y}\right)}{\partial x} - \frac{\partial \left(x^2e^{2y}\right)}{\partial y} \right) \\ &= 2 \cdot 10^4 \left(2xe^{2y} - 2xe^{2y} \right) = 0 \end{aligned}$$

For all field points (x, y). The field is conservative. Consequently, we have two "labor saving" ways we can determine the value of $W = \int_C \mathbf{F} \cdot \mathbf{d}r$: choose a simpler path for the line integral, or build the potential function for the conservative vector field, and evaluate the difference of potential between the end point and the beginning point. We develop each alternative.

<u>Alternative 1:</u> Choose a simpler path.

Take the new (oriented) curve C_2 to be the straight line segment along the *x* axis from $(0, 0)$ to $(12\pi, 0)$. Because the vector field is conservative, line integrals of it are *independent* of path. So,

$$\boxed{\int_C \mathbf{F} \cdot \mathbf{d}r = W = \int_{C_2} \mathbf{F} \cdot \mathbf{d}r}$$

We build a parameterized line integral for $\mathbf{F}(x, y)$ over C_2.

1) Move over the curve.

The parameterization is straight forward: $x(t)=t,\ \ y(t)=0,\ \ 0 \le t \le 12\pi$.

2) Build the vector line element

$$d\mathbf{r}(t)=\mathbf{v}(t)\,dt=(1\mathbf{i}+0\mathbf{j})\,dt$$

3) Specify the vector in the field encountered at each instant in the motion

$$\begin{aligned}\mathbf{G}(t)=\mathbf{F}(x(t),y(t))&=\mathbf{F}(t,0)\\ &=2\cdot 10^4\left(te^0\mathbf{i}+t^2e^0\mathbf{j}\right)=2\cdot 10^4\left(t\mathbf{i}+t^2\mathbf{j}\right)\end{aligned}$$

4) Compute the decrement of work done over as line element, $\boxed{\mathbf{F}\cdot\mathbf{d}r}$

For this motion, at each instant *t*, the decrement of work is

$$\begin{aligned}\mathbf{F}\cdot d\mathbf{r}=\mathbf{G}(t)\cdot\mathbf{v}(t)\,dt&=2\cdot 10^4\left(t\mathbf{i}+t^2\mathbf{j}\right)\cdot(1\mathbf{i}+0\mathbf{j})\,dt\\ &=2\cdot 10^4\,t\,dt\end{aligned}$$

5) Integrate over the extent of the motion

Integrating the decrement of work over the extent of the motion produces the parameterized integral

$$\boxed{\int_C \mathbf{F}\cdot\mathbf{d}r=W=\int_{C_2}\mathbf{F}\cdot\mathbf{d}r=2\cdot 10^4\int_{t=0}^{12\pi} t\,dt}$$

6) Evaluate the parameterized integral

Straightforwardly,

$$\boxed{\begin{aligned}&\int_C \mathbf{F}\cdot\mathbf{d}r=W=\int_{C_2}\mathbf{F}\cdot\mathbf{d}r=2\cdot 10^4\int_{t=0}^{12\pi} t\,dt=10^4(12\pi)^2\\ &\int_C \mathbf{F}\cdot\mathbf{d}r=W=144\pi^2\cdot 10^4\ \ (eV)\end{aligned}}$$

<u>Alternative 2:</u> Build and use the potential function

We seek $f(x,y)$ for which

$$\boxed{\nabla f(x,y)=\frac{\partial f(x,y)}{\partial x}\mathbf{i}+\frac{\partial f(x,y)}{\partial x}\mathbf{j}=\mathbf{F}(x,y)=q\mathbf{E}(x,y)=2\cdot 10^4\left(xe^{2y}\mathbf{i}+x^2e^{2y}\mathbf{j}\right)}$$

Examining ***i*** component, we deduce

$$\begin{aligned}\frac{\partial f(x,y)}{\partial x}&=2\cdot 10^4\,xe^{2y}\\ f(x,y)&=10^4x^2e^{2y}+H(y)\end{aligned}$$

where $H(y)$ is a function solely of the variable y, and is to be determined. Differentiating this form for $f(x,y)$ with respect to y, and equating the result to the ***j*** component of the force vector field, we obtain a one-parameter family of potential functions for the force vector field:

$$\boxed{\begin{aligned} 2\cdot 10^4 xe^{2y} + \frac{d(H(y))}{dy} &= \frac{\partial f(x,y)}{\partial y} = 2\cdot 10^4 xe^{2y} \\ \frac{d(H(y))}{dy} &= K, \quad \text{(a constant)} \\ f(x,y) &= 10^4 x^2 e^{2y} + K \end{aligned}}$$

By the "fundamental theorem of line integrals" for vector fields,

$$\boxed{\begin{aligned} W &= \int_C \mathbf{F}\cdot \mathbf{d}r = f(12\pi,0) - f(0,0) \\ &= \left(10^4 (12\pi)^2 e^{2\cdot 0} + k\right) - \left(10^4 \cdot 0^2 e^{2\cdot 0} + K\right) = 10^4 (12\pi)^2 \\ \int_C \mathbf{F}\cdot\mathbf{d}r &= W = 144\pi^2 \cdot 10^4 \ \ (eV) \end{aligned}}$$

<u>Alternative 3:</u> Integrate over the original trajectory.
As is apparent, this alternative is the least attractive of the three. For completeness, we specify how we can move over the trajectory. We leave it to the reader to complete setting up the line integral from that point

1) Move over the trajectory
As we are told, the trajectory (a spiral) is characterized in polar coordinates by $r(\theta) = \theta$. The extent of the segment of the spiral we can deduce from the coordinates of the beginning and end points, viewed in polar coordinates: $0 \le \theta \le 12\pi$.

Consequently, if we "go polar", and we let the angle be the "ticking clock"; e.g. $\theta = t$, a parameterization for the trajectory and its position vector function becomes

$$\boxed{\begin{aligned} x(t) &= r(t)\cos(t) \underbrace{=}_{r=\theta} t\cos(t) \\ y(t) &= r(t)\sin(t) \underbrace{=}_{r=\theta} t\sin(t) \\ \mathbf{r}(t) &= x(t)\mathbf{i} + y(t)\mathbf{j} = t\cos(t)\mathbf{i} + t\sin(t)\mathbf{j}, \ \ 0 \le t \le 2\pi \end{aligned}}$$

From this point the computation follows the line of reasoning set forth in alternative 1 of problem 2.

2) Build the vector line element
The velocity associated with this motion is

$$\mathbf{v}(t) = \frac{d}{dt}(\mathbf{r}(t)) = (\cos(t) - t\sin(t))\mathbf{i} + (\sin(t) + t\cos(t))\mathbf{j},\ \ 0 \le t \le 12\pi$$

The vector line element is

$$d\mathbf{r}(t) = \mathbf{v}(t)\,dt = \left[(\cos(t) - t\sin(t))\mathbf{i} + (\sin(t) + t\cos(t))\mathbf{j}\ \right]dt,\ \ 0 \le t \le 12\pi$$

3) Specify the vector in the field encountered at each instant in the motion

$$\text{For } \mathbf{F}(x,y) = 2\cdot 10^4 e^{2y}\left(x\mathbf{i} + x^2\mathbf{j}\right),$$

$$\begin{aligned} \mathbf{G}(t) = \mathbf{F}(x(t), y(t)) &= \mathbf{F}(t\cos(t), t\sin(t)) \\ &= 2\cdot 10^4\left(t\cos(t)e^{2t\sin(t)}\mathbf{i} + t^2\cos^2(t)e^{2t\sin(t)}\mathbf{j}\right) \end{aligned}$$

4) Compute the decrement of work done over as line element, $\boxed{\mathbf{F}\cdot\mathbf{d}r}$
For this motion, at each instant *t*, $0 \le t \le 12\pi$ the decrement of work is

$$\begin{aligned} \mathbf{F}\cdot d\mathbf{r} &= \mathbf{G}(t)\cdot\mathbf{v}(t)\,dt \\ &= 2\cdot 10^4\left(t\cos(t)e^{2t\sin(t)}\mathbf{i} + t^2\cos^2(t)e^{2t\sin(t)}\mathbf{j}\right)\cdot\left[(\cos(t) - t\sin(t))\mathbf{i} + (\sin(t) + t\cos(t))\mathbf{j}\ \right]dt \\ &= 2\cdot 10^4\begin{pmatrix} t\cos^2(t)e^{2t\sin(t)} - t^2\cos(t)\sin(t)e^{2t\sin(t)} + \\ t^2\cos^2(t)\sin(t)e^{2t\sin(t)} + t^3\cos^3(t)e^{2t\sin(t)} \end{pmatrix}dt \\ &= 2\cdot 10^4 e^{2t\sin(t)}\left(t^3\cos^3(t) + t^2\left(\cos^2(t)\sin(t) - \cos(t)\sin(t)\right) + t\cos^2(t)\right)dt \end{aligned}$$

5) Integrate over the extent of the motion
Integrating the decrement of work over the extent of the motion produces the parameterized integral

$$\boxed{W = \int_C \mathbf{F}\cdot\mathbf{d}r = 2\cdot 10^4\int_{t=0}^{12\pi} e^{2t\sin(t)}\left(t^3\cos^3(t) + t^2\left(\cos^2(t)\sin(t) - \cos(t)\sin(t)\right) + t\cos^2(t)\right)dt}$$

6) Evaluate the parameterized integral
Here, we turn to the Voyage 200. In fact, in "exact" mode, the Voyage200 returns the representation of the integral to be computed, asserting that there is no

quadrature or "anti-derivative" that can be evaluated at end points. If we evaluate the integral numerically ("diamond enter"), we obtain (after an impressive interval of time)

$$\boxed{W = \int_C \mathbf{F} \cdot \mathbf{d}r \cong 3.10439 \times 10^{23}}$$

Ed note: In contrast, Maple evaluates the integral using complex integration. It returns the exact value

$$\boxed{W = \int_C \mathbf{F} \cdot \mathbf{d}r = 144\pi^2 \cdot 10^4 \ \ (eV)}$$

I conjecture that the numerical integration procedure used by the Voyage 200 is inappropriate for the exponential growth in the integrand.

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3. You create a reflecting surface as that segment of the upper hemisphere of the sphere $x^2 + y^2 + z^2 = 25$ that is contained in the cone $3z^2 = x^2 + y^2$, as shown in the figure 'Question 3'. The unit of measure is centimeters (cm)

a) What is the area of the surface of the reflector $\left(\text{in } cm^2\right)$?

b) The point $(0,3,4)$ lies on the reflector. What is the equation of the plane tangent to the surface at that point?

Solution

By a multi-step process, we move on the segment of the surface, then create from the movement a surface normal and area element. These two mathematical objects will give us the capability to resolve the two questions of interest.

Move on the surface, restricting the extent of movement so as to produce the segment of interest

We perceive three possible ways of moving on the surface of interest: a "motion by graph", a motion that takes advantage of cylindrical coordinate representation of spheres and cylinders, and a motion that takes advantage of the spherical coordinate representation of spheres and cylinders. We choose the middle alternative.

In cylindrical coordinates, the equations for the hemisphere and the bounding cylinder are, respectively $\left(\text{recall } x^2 + y^2 = r^2\right)$,

$$z = \sqrt{25 - r^2} \quad (z \geq 0)$$

$$\sqrt{3}\, z = r$$

Simultaneously solving the equations gives us an upper bound on the radial variable:

$$\begin{aligned}
\frac{r^2}{3} &= 25 - r^2 \\
4r^2 &= 75 \\
r &= \frac{\sqrt{75}}{2} = \frac{5}{2}\sqrt{3}
\end{aligned}$$

The motion on the hemispherical surface arises when we "go polar" in the x and y coordinates, then "lift" to the surface using the equation for the hemisphere to characterize the z coordinate:

$$\begin{aligned}
x(r,\theta) &= r\cos(\theta) \\
y(r,\theta) &= r\sin(\theta) \\
z(r,\theta) &= \sqrt{25 - r^2}
\end{aligned}$$

The extent of the motion is apparent (see the figure given). The "shadow" of the segment of the hemisphere in the x-y plane is a disk whose radius is the upper bound established above. Consequently, the ranges of the parameters are

$$\begin{aligned}
&0 \le r \le \frac{5}{2}\sqrt{3} \\
&0 \le \theta \le 2\pi
\end{aligned}$$

In sum, our motion can be characterized by the two-parameter position vector function

$$\boxed{\begin{aligned}
&\mathbf{r}(r,\theta) = r\cos(\theta)\mathbf{i} + r\sin(\theta)\mathbf{j} + \sqrt{25 - r^2}\,\mathbf{k}, \\
&0 \le r \le \frac{5}{2}\sqrt{3}, \quad 0 \le \theta \le 2\pi
\end{aligned}}$$

Build the vector and scalar surface area elements

From this point on, the process becomes quite mechanical. First, we build the two velocities associated with the motion,

$$\begin{aligned}
\mathbf{v}_1(r,\theta) &= \frac{\partial \mathbf{r}(r,\theta)}{\partial r} = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j} - \frac{r}{\sqrt{25 - r^2}}\mathbf{k} \\
\mathbf{v}_2(r,\theta) &= \frac{\partial \mathbf{r}(r,\theta)}{\partial \theta} = -r\sin(\theta)\mathbf{i} + r\cos(\theta)\mathbf{j} + 0\mathbf{k}
\end{aligned}$$

The surface normal we build from the cross product of these vector functions

$$\mathbf{N}(r,\theta)=\mathbf{v}_1(r,\theta)\times\mathbf{v}_2(r,\theta)=\det\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos(\theta) & \sin(\theta) & -\frac{r}{\sqrt{25-r^2}}\\ -r\sin(\theta) & r\cos(\theta) & 0\end{bmatrix}$$

$$=\frac{r^2\cos(\theta)}{\sqrt{25-r^2}}\mathbf{i}+\frac{r^2\sin(\theta)}{\sqrt{25-r^2}}\mathbf{j}+r\left(\cos^2(\theta)+\sin^2(\theta)\right)\mathbf{k}$$

$$=\frac{r^2\cos(\theta)}{\sqrt{25-r^2}}\mathbf{i}+\frac{r^2\sin(\theta)}{\sqrt{25-r^2}}\mathbf{j}+r\,\mathbf{k}$$

The vector and scalar surface elements follow:

$$d\mathbf{S}(r,\theta)=\mathbf{N}(r,\theta)drd\theta=\left(\frac{r^2\cos(\theta)}{\sqrt{25-r^2}}\mathbf{i}+\frac{r^2\sin(\theta)}{\sqrt{25-r^2}}\mathbf{j}+r\,\mathbf{k}\right)drd\theta$$

$$dS(r,\theta)=\left\|d\mathbf{S}(r,\theta)\right\|=\left\|\mathbf{N}(r,\theta)\right\|drd\theta=\sqrt{\left(\frac{r^2\cos(\theta)}{\sqrt{25-r^2}}\right)^2+\left(\frac{r^2\sin(\theta)}{\sqrt{25-r^2}}\right)^2+r^2}\;drd\theta$$

$$dS(r,\theta)=\left\|d\mathbf{S}(r,\theta)\right\|=\left\|\mathbf{N}(r,\theta)\right\|drd\theta=\sqrt{\frac{r^4}{25-r^2}+r^2}\;drd\theta=\sqrt{\frac{25r^2}{25-r^2}}\;drd\theta$$

$$dS(r,\theta)=\frac{5r}{\sqrt{25-r^2}}drd\theta$$

Resolve the questions using dS and ***N***

Now we have the tools with which to resolve the questions of interest.

a) The value of the surface area integral we can construct as the parameterized double integral of the scalar surface area element associated with our motion over the disk *D* determined by the limits of the parameters,

$$\iint_S dS=\iint_D\frac{5r}{\sqrt{25-r^2}}drd\theta=\int_{\theta=0}^{2\pi}\int_{r=0}^{5\sqrt{3}/2}\frac{5r}{\sqrt{25-r^2}}\,drd\theta$$

The evaluation is straightforward:

$$
\begin{aligned}
\iint_S dS &= \int_{\theta=0}^{2\pi}\int_{r=0}^{5\sqrt{3}/2} \frac{5r}{\sqrt{25-r^2}}\,dr d\theta \\
&= 5\int_{\theta=0}^{2\pi}\left[-\sqrt{25-r^2}\right]_{r=0}^{r=5\sqrt{3}/2} d\theta = 5\int_{\theta=0}^{2\pi}\left[5-\frac{5}{2}\right]d\theta \\
\iint_S dS &= \frac{25}{2}\int_{\theta=0}^{2\pi} d\theta = 25\pi
\end{aligned}
$$

b) To compute the equation of the tangent plane, we must determine the values of the parameters that locate us at the point $(0,3,4)$. We use the parameterized equations for the motion:

$$
\begin{aligned}
0 &= x(r,\theta) = r\cos(\theta) \\
3 &= y(r,\theta) = r\sin(\theta) \\
4 &= z(r,\theta) = \sqrt{25-r^2}
\end{aligned}
$$

We deduce $r = 3,\ \theta = 0\,(rads)$. The surface normal vector at this location is

$$
\mathbf{N}(3,0) = \frac{3^2\cos(0)}{\sqrt{25-3^2}}\mathbf{i} + \frac{3^2\sin(0)}{\sqrt{25-3^2}}\mathbf{j} + 3\mathbf{k} = \frac{9}{4}\mathbf{i} + 3\mathbf{k}
$$

Let $P_0(0,3,4)$ and $R(x,y,z)$ denote the "anchor point" and a generic point on the plane tangent to the surface S at P_0. Then the vector equation that specifies that the plane is perpendicular to the normal vector produces the equation of the tangent plane:

$$
\begin{aligned}
\mathbf{N}(3,0)\cdot\overrightarrow{P_0R} &= 0 \\
\left(\frac{9}{4}\mathbf{i}+3\mathbf{k}\right)\cdot\left((x-0)\mathbf{i}+(y-3)\mathbf{j}+(z-4)\mathbf{k}\right) &= 0 \\
\frac{9}{4}x + 3(z-4) &= 0 \\
9x + 12z &= 48
\end{aligned}
$$

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4. Evaluate the line integral

$$
\int_C y^2 dx + 2x^2 dy
$$

Over the curve C that runs along the quarter circle $x^2+y^2=4$ clockwise from $(0,2)$ to $(2,0)$, along the x axis from $(2,0)$ to the origin, then up the y axis from the origin to the original starting point $(0,2)$, which is depicted in the figure 'Question 4'.

Solution

Determine the choices Green's theorem provides in this case

The integral presented to us is in Pfaffian format. To apply Green's Theorem it is advantageous to write the integral in vector format. In the plane, set

$$\mathbf{F}(x,y)=P(x,y)\mathbf{i}+Q(x,y)\mathbf{j}=y^2\mathbf{i}+2x^2\mathbf{j}$$

then

$$\int_C y^2dx+2x^2dy=\int_C \mathbf{F}\cdot d\mathbf{r}$$

Let $\tilde{C}=-C$ denote the curve whose locus of points is the same as that of C, but whose orientation is opposite (counterclockwise). Because the vector field can sense orientation, the value of the line integral of the vector field over C and $\tilde{C}$ are related:

$$\int_{\tilde{C}=-C} \mathbf{F}\cdot d\mathbf{r}=-\int_C \mathbf{F}\cdot d\mathbf{r}$$

Because the curve $\tilde{C}$ is simple and closed, Green's theorem provides a choice between computing a line integral over it and computing the flux integral of the curl of the vector field through the surface D in the plane that the curve encloses, provided the surface is oriented consistently with the orientation of $\tilde{C}$. In this case the consistent orientation for D is $\mathbf{n}=\mathbf{k}$:

$$\int_{\tilde{C}=\partial D} \mathbf{F}\cdot d\mathbf{r}=\iint_D curl\mathbf{F}\cdot\mathbf{n}dA=\iint_D\left(\frac{\partial Q(x,y)}{\partial x}-\frac{\partial P(x,y)}{\partial y}\right)dA$$

To make a choice as to which integral to set up and evaluate, we compute $curl\mathbf{F}$:

$$curl\mathbf{F}(x,y)=\left(\frac{\partial Q(x,y)}{\partial x}-\frac{\partial P(x,y)}{\partial y}\right)\mathbf{k}=(4x-2y)\mathbf{k}$$

we choose to go with the surface integral.

Set up and evaluate the integral of choice

We have

$$\iint_D curl\mathbf{F}\cdot\mathbf{n}dA=\iint_D(4x-2y)\,dA$$

where, from the figure, we see that D is the ¼ disk based at $(0,0)$ of radius 2. The geometry of the domain of integration and the simplicity of the integrand suggest that we "go polar" with the double integral. We set

$$\begin{aligned} &x = r\cos(\theta) \\ &y = r\sin(\theta) \\ &0 \le r \le 2,\ \ 0 \le \theta \le \pi/4 \end{aligned}$$

We take $dA = rdrd\theta$. The double integral becomes the iteration of integrals

$$\iint_D (4x-2y)\,dA = \int_{\theta=0}^{\pi/4}\int_{r=0}^{2}\left(4r\cos(\theta)-2r\sin(\theta)\right)rdrd\theta$$

Evaluating the integral is straightforward.

$$\begin{aligned}
\iint_D (4x-2y)\,dA &= \int_{\theta=0}^{\pi/4}\int_{r=0}^{2}\left(4r\cos(\theta)-2r\sin(\theta)\right)rdrd\theta \\
&= \int_{\theta=0}^{\pi/4}\left[\frac{4r^3}{3}\cos(\theta)-\frac{2r^3}{3}\sin(\theta)\right]_{r=0}^{2} d\theta \\
&= \int_{\theta=0}^{\pi/4}\left[\frac{32}{3}\cos(\theta)-\frac{16}{3}\sin(\theta)\right]d\theta \\
&= \left[\frac{32}{3}\sin(\theta)+\frac{16}{3}\cos(\theta)\right]_{\theta=0}^{\pi/4} \\
\iint_D (4x-2y)\,dA &= 8\sqrt{2}+\frac{16}{3}
\end{aligned}$$

In conclusion, the value of the line integral of interest is

$$\boxed{\begin{aligned} &\int_C y^2dx+2x^2dy = \int_C \mathbf{F}\cdot d\mathbf{r} = -\int_{\tilde{C}=-C}\mathbf{F}\cdot d\mathbf{r} \underbrace{=}_{\text{Green's Thm}} -\iint_D (4x-2y)\,dA = -\left(8\sqrt{2}-\frac{16}{3}\right) \\ &\qquad = \frac{16}{3}-8\sqrt{3} \cong -8.52 \end{aligned}}$$

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5. At 0900, the temperature in CH334 varies with location in the room, and may be modeled by the scalar field (real-valued function)

$$T(x,y,z) = x^2+y^2-z^2$$

The variation in temperature produces a *heat flux* vector field, defined by

$$\boldsymbol{F}(x,y,z) = -\nabla T(x,y,z)$$

a) Specify the heat flux vector field in terms of its components.
b) Is the heat flux vector field a "curl free" vector field? Justify your assertion by analysis or computation.
c) Is the heat flux vector field a "divergence free" vector field? Justify your assertion by analysis or computation.

Solution

a) The heat flux vector field is straightforward:

$$\boxed{\begin{aligned}\mathbf{F}(x,y,z) = -\nabla T(x,y,z) &= -\left(\frac{\partial\left(x^2+y^2-z^2\right)}{\partial x}\right)\mathbf{i} - \left(\frac{\partial\left(x^2+y^2-z^2\right)}{\partial y}\right)\mathbf{j} - \left(\frac{\partial\left(x^2+y^2-z^2\right)}{\partial z}\right)\mathbf{k} \\ &= -2x\mathbf{i} - 2y\mathbf{j} + 2z\mathbf{k}\end{aligned}}$$

b) We seek to determine if $curl\mathbf{F}(x,y,z)$ is the zero vector field. Because **F** is a *gradient* vector field and its components are continuously differentiable, it follows by vector field identity,

$$\boxed{curl(grad\ f) = \mathbf{0}}$$

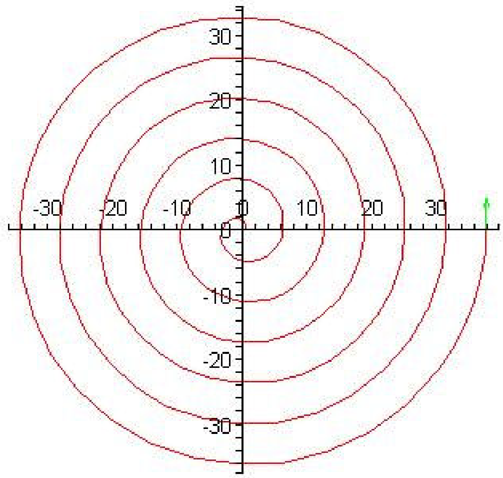
that the vector field is ***curl free.***

c) We compute the divergence of the heat flux vector field. Recall that $div(grad\ f) = \nabla^2 f$ is the *Laplacian* of the scalar function $f(x,y,z)$

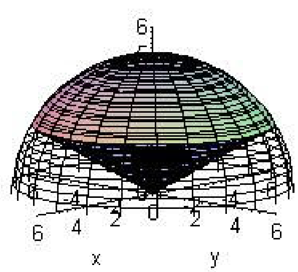
$$\boxed{-\nabla^2 T(x,y,z) = div\mathbf{F}(x,y,z) = \frac{\partial(2x)}{\partial x} - \frac{\partial(2y)}{\partial y} + \frac{\partial(2z)}{\partial z} = 2}$$

■

Question 2



Question 3



Question 4