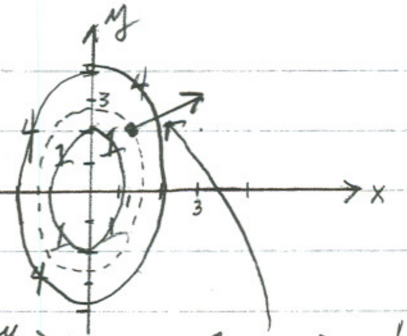
SM221 CALC III FALL 2009 PRACTICE II SOLUTIONS

1. (a) $f(x,y) = x^2 + \frac{y^2}{4}$

$x^2 + \frac{y^2}{4} = 1 \Leftrightarrow x^2 + \frac{y^2}{2^2} = 1$

both ellipses

$x^2 + \frac{y^2}{4} = 4 \Leftrightarrow \frac{x^2}{2^2} + \frac{y^2}{4^2} = 1$



(b) $\vec{\nabla} f(1,2) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \Big|_{(1,2)} = \left\langle 2x, \frac{2y}{4} \right\rangle \Big|_{(1,2)} = \langle 2, 1 \rangle$ ⊥ contour curve thru $(1,2)$.

(c) $D_{\hat{u}} f(1,2) = \vec{\nabla} f(1,2) \cdot \hat{u} = \langle 2,1 \rangle \cdot \frac{\langle 3,4 \rangle}{\sqrt{9+16}} = \frac{6+4}{5} = 2$

(d) max dir deriv at $(1,2) = |\vec{\nabla} f(1,2)| = |\langle 2,1 \rangle| = \sqrt{5}$

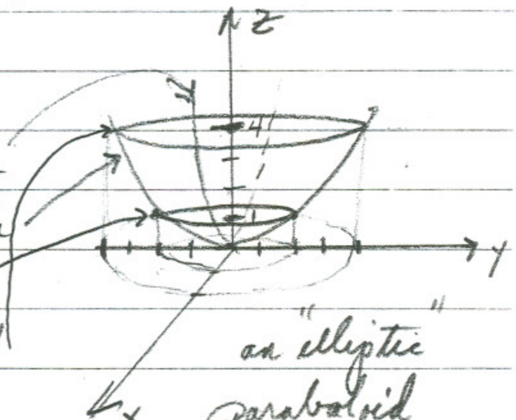
(e) perpendicular to $\vec{\nabla} f(1,2) \Rightarrow \hat{u} = \pm \frac{\langle 1,-2 \rangle}{\sqrt{5}}$

(f) x-z trace $\Rightarrow y = 0 \Rightarrow z = x^2 + \cancel{y^2/4}^{\,0} \Rightarrow z = x^2$ parabola

y-z trace $\Rightarrow x = 0 \Rightarrow z = \cancel{x^2} + \frac{y^2}{4} \Rightarrow z = \frac{y^2}{4}$ parabola

$z = 1 \Rightarrow 1 = x^2 + y^2/4$ ellipse up to height of 1

$z = 4 \Rightarrow 4 = x^2 + y^2/4 \Rightarrow 1 = \frac{x^2}{2^2} + \frac{y^2}{4^2}$ ellipse at height 4



an "elliptic" paraboloid

2. (a) $\frac{\partial}{\partial x}(y^2 e^{3x}) = 3y^2 e^{3x}$; $\frac{\partial}{\partial y}(y^2 e^{3x}) = 2y e^{3x}$

(b) $\frac{\partial}{\partial x}\left((3xy + 2x)^5\right) = 5(3xy+2x)^4 \cdot (3y+2)$; $\frac{\partial}{\partial y}\left((3xy+2x)^5\right) = 5(3xy+2x)^4 (3x)$

(c) $\frac{\partial}{\partial x}\left(\frac{y^2}{x+1}\right) = \frac{0 - y^2 \cdot 1}{(x+1)^2} = \frac{-y^2}{(x+1)^2}$; $\frac{\partial}{\partial y}\left(\frac{y^2}{x+1}\right) = \frac{2y(x+1) - y^2 \cdot 0}{(x+1)^2} = \frac{2y}{x+1}$

3. (a) $\frac{\partial f}{\partial x}(20,10) \doteq \frac{\Delta f}{\Delta x} = \frac{f(30,10) - f(10,10)}{30-10} = \frac{36-16}{20} = 1$

(b) $\frac{\partial f}{\partial y}(20,10) \doteq \frac{\Delta f}{\Delta y} = \frac{f(20,20) - f(20,0)}{20-0} = \frac{21-26}{20} = -.25$

(c) $z - z_0 = m(x-x_0) + n(y-y_0)$ tangent plane

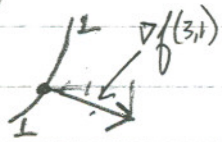
$z - 25 = 1(x-20) - .25(y-10)$

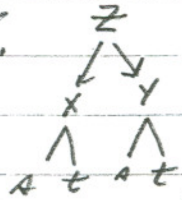
(d) use $z = 25 + 1(x-20) - .25(y-10)$ at $(x,y) = (21,9)$ ("linearization")

$\Rightarrow z = 25 + 1(21-20) - .25(9-10) = 25 + 1 + .25 = 26.25$

4. (a) $f_x(3,1) \doteq \dfrac{\overset{2}{f(3.5,1)} - \overset{0}{f(2.5,1)}}{3.5-2.5} = 2$ (b) $f_y(3,1) \doteq \dfrac{\overset{0}{f(3,2.5)} - \overset{2}{f(3,.5)}}{2.5-.5} = \dfrac{-2}{2} = -1$

(c) $\vec{\nabla} f(3,1) = \langle f_x, f_y \rangle \big|_{(3,1)} \doteq \langle 2, -1 \rangle$; $\perp$ level curve going through (3,1).



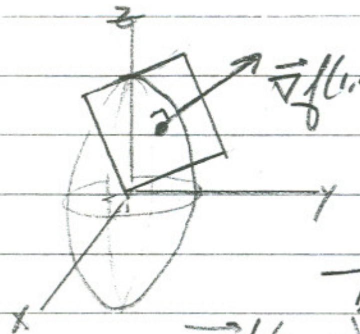
5. 

$\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s}$ $z = e^{x-3y}$; $x = s^2 t$
$y = st^2$

$= \left(e^{x-3y}\right)(2st) + \left(e^{x-3y}\right)(-3)(t^2)$

(at $s=2, t=1$; $x = 2^2\cdot 1 = 4$; $y = 2\cdot 1^2 = 2$) $= e^{(4-3\cdot 2)}(2\cdot 2\cdot 1) + e^{(4-3\cdot 2)}(-3)\cdot 1^2$

$= e^{-2}(4) + e^{-2}(-3) = e^{-2} = \dfrac{1}{e^2}$

6. 

$\vec{\nabla} f(1,2,3) = \vec{N}_\perp$: The ellipsoid $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 3$ can be thought of as a level surface for $f(x,y,z) = x^2 + y^2/4 + z^2/9$. So where $f = 3$.

The pt (1,2,3) is on that surface, so the gradient $\vec{\nabla} f(1,2,3)$ is $\perp$ the surface. It is also $\perp$ to the tangent plane and in the direction of the normal line.

$\vec{\nabla} f(1,2,3) = \langle f_x, f_y, f_z \rangle \big|_{(1,2,3)} = \langle 2x, y/2, \frac{2z}{9} \rangle \big|_{(1,2,3)} = \langle 2, 1, 2/3 \rangle$

$\Rightarrow$ tangent plane: $2(x-1) + 1(y-2) + 2/3(z-3) = 0$

$\Rightarrow$ normal line: $x = 1 + 2t$; $y = 2 + 1t$; $z = 3 + 2/3\, t$.

7. $f(x,y) = y^2 + 6x^3 + 18x^2 - 6xy + 18x - 18y$.

$f_x(x,y) = 18x^2 + 36x - 6y + 18 = 0$

$f_y(x,y) = 2y - 6x - 18 = 0 \Rightarrow y = 3x + 9 \Rightarrow 18x^2 + 36x - 6(3x+9) + 18 = 0$

$\Rightarrow 18x^2 + 18x - 36 = 0$

$\Rightarrow x^2 + x - 2 = 0$

$(x+2)(x-1) = 0 \Rightarrow x = 1$ or -2

$\Rightarrow y = 12$ or 3

so (1,12) and (−2,3) are critical pts.

$D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - \left[f_{xy}(x,y)\right]^2$

$= (36x + 36)(2) - 36$

at (1, 12); $D(1,2) = 144 - 36 > 0$ and $f_{xx}(1,2) = 72 > 0 \Rightarrow (1,12)$ is a local min.

at (−2, 3); $D(-2,3) = (-36)(2) - 36 = -108 \Rightarrow$ neither a local max nor min