SM221 CALCULUS III FALL 2009 PRACTICE PROBLEMS III SOLUTIONS

1. (a) $\text{Vol} = \int_0^{40}\int_0^{40} f(x,y)\,dy\,dx$

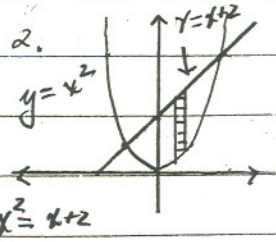
$\doteq (\Delta x)(\Delta y)\left[f(10,10)+f(30,10)+f(10,30)+f(30,30)\right]$

$= (20)(20)\left[2 + 6 + 3 + 3\right]$

$= 400 \cdot 14 = 5{,}600 \text{ ft}^3$

(b) ave = vol/area = 5,600/1600 = 3.5 ft

2.



$\text{mass} = \iint_R \rho\,dA$

$= \int_{-1}^{2}\int_{x^2}^{x+2} y\,dy\,dx$ (dxdy requires 2 integrals)

$= \int_{-1}^{2}\left[\frac{y^2}{2}\Big|_{x^2}^{x+2}\right]dx$

$x^2 = x+2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

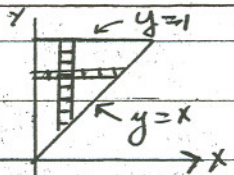
$x = -1, 2$

$= \int_{-1}^{2}\left[\frac{(x+2)^2}{2} - \frac{x^4}{2}\right]dx = \frac{(x+2)^3}{6} - \frac{x^5}{10}\Big|_{-1}^{2}$

$= \left(\frac{4^3}{6} - \frac{2^5}{10}\right) - \left(\frac{(1)^3}{6} - \frac{(-1)^5}{10}\right) = \frac{(64-1)}{6} - \frac{(2^5+1)}{10}$

$= \frac{63}{6} - \frac{33}{10} = 10.5 - 3.3 = 7.2 = 36/5$

3. $\int_0^1\int_x^1 \cos(y^2)\,dy\,dx$ ← top, ← bottom



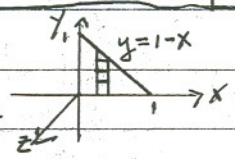
$= \int_{y=0}^{1}\int_{x=0}^{x=y} \cos(y^2)\,dx\,dy$ ← right, ← left

$= \int_0^1 \left[x\cos(y^2)\Big|_{x=0}^{y}\right]dy$

$= \int_0^1 \cos(y^2)\,y\,dy$ u-subst let $u = y^2$

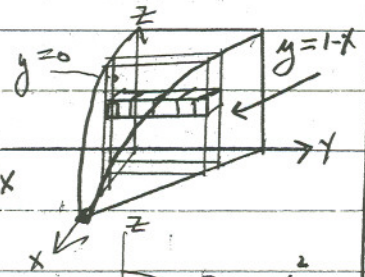
$= \frac{\sin(y^2)}{2}\Big|_0^1 = \frac{\sin(1)}{2} \doteq .4207$

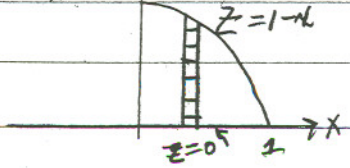
4. (a) $\text{Vol} = \int_{x=0}^{1}\int_{y=0}^{1-x} \underbrace{(1-x^2)}_{\text{height to "ceiling"}}\,\underbrace{dy\,dx}_{\text{base area}}$



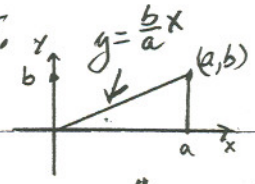
(b) $\text{Vol} = \iiint_E \underbrace{1}_{f(x,y,z)}\,dV$

$= \int_{x=0}^{1}\int_{z=0}^{1-x^2}\int_{y=0}^{1-x} 1\,dy\,dz\,dx$





5.



$\rho = k$ (const)

$\Rightarrow \text{mass} = \iint_R \rho\,dA = \iint_R k\,dA$

$= k\iint_R dA = k\,\text{area} = k\,ab/2$

$M_y = \Sigma\Sigma$ "x-moment" $= \iint_R x\rho\,dA$

$= \int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}x} x\,k\,y\,dx = \int_0^a\left[k\,x\,y\Big|_{y=0}^{\frac{b}{a}x}\right]dx$

$= \int_0^a \frac{kb}{a}x^2\,dx = \frac{kb}{a}\frac{x^3}{3}\Big|_0^a = kba^2/3$

so $\bar{x} = \frac{M_y}{m} = \frac{kba^2/3}{kab/2} = \frac{2}{3}a$.

show $\bar{y} = \frac{1}{3}b$. So $(\bar{x},\bar{y}) = \left(\frac{2}{3}a, \frac{1}{3}b\right)$.

make sense? why?

6.

$\rho(x,y) = kd = k\sqrt{x^2+y^2}$

$\text{mass} = \iint_R \rho\,dA$

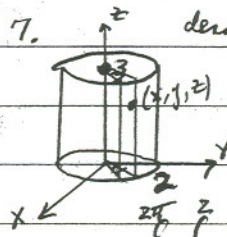
use polar coords

$dA = \square\,dr$ ($r\,d\theta$)

$= \int_0^{\pi}\int_{r=1}^{2} (kr)\,r\,dr\,d\theta$

$= \int_0^{\pi}\left[\int_1^2 kr^2\,dr\right]d\theta = \int_0^{\pi} k\frac{7}{3}\,d\theta = \frac{7}{3}\pi k$.

7.



dens: $\rho(x,y,z) = z$ (dist from plane x-y)

$\bar{x} = \bar{y} = 0$ from symmetry

$\text{mass} = \iiint_E \rho\,dV$

$= \int_0^{2\pi}\int_{r=0}^{2}\int_{z=0}^{3} z\,(r\,dz\,dr\,d\theta) = \int_0^{2\pi}\int_0^2 \frac{9}{2}r\,dr\,d\theta$

$= \int_0^{2\pi} 9\,d\theta = 18\pi$.

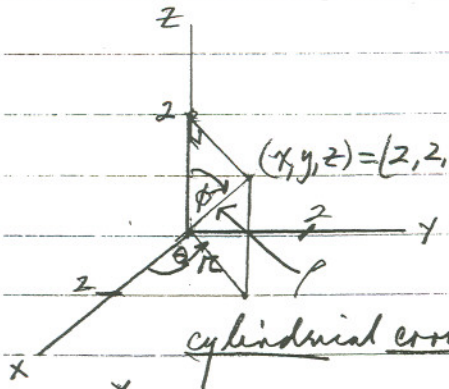
$M_{xy} = \iiint_E z\,\rho\,dV = \int_0^{2\pi}\int_0^2\int_0^3 z\cdot z\,r\,dz\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^2 9r\,dr\,d\theta = \int_0^{2\pi} 18\,d\theta = 36\pi$

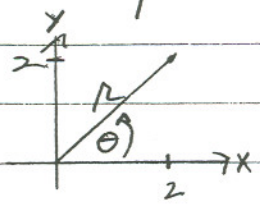
so $\bar{z} = M_{xy}/m = \frac{36\pi}{18} = 2$ make sense?

why not $\bar{z} = 1.5$ halfway up?

8. (a) $(x,y,z) = (2,2,2)$



cylindrical coords: (r,θ,z)



$r = \sqrt{x^2+y^2} = \sqrt{2^2+2^2}$
$= \sqrt{8} = 2\sqrt{2}$

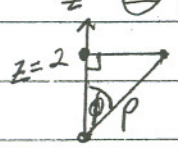
$\tan(\theta) = y/x = 2/2$
$\Rightarrow \theta = \pi/4$

so $(r,\theta,z) = (2\sqrt{2}, \pi/4, 2)$

spherical coords: (ρ,θ,ϕ)

$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{3\cdot 2^2} = 2\sqrt{3}$

$\theta = \pi/4$



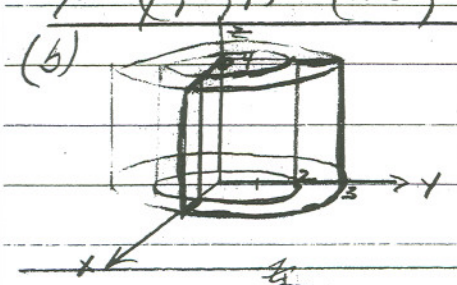
$\frac{z}{\rho} = \cos(\phi)$, $z = \rho\cos(\phi)$

$\frac{2}{2\sqrt{3}} = \cos(\phi) \Rightarrow \phi = \cos^{-1}\left(\sqrt{3}/3\right)$

$\Rightarrow \phi = .955$ rads.

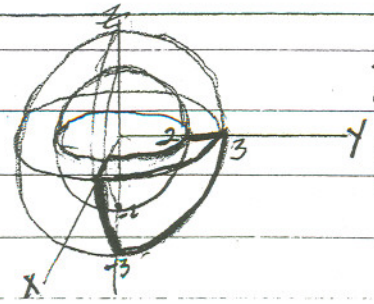
so $(\rho,\theta,\phi) = (2\sqrt{3}, \pi/4, .955)$

(b)



$2 \le r \le 3$
$0 \le \theta \le \pi/2$
$0 \le z \le 4$

(c)



$2 \le \rho \le 3$
$0 \le \theta \le \pi/2$
$\frac{\pi}{2} \le \phi \le \pi$

9. Assume $\iint_D f(x,y)\,dA$ exists.
(property #11). If $m \le f(x,y) \le M$ for all (x,y) in D, then

$$m\,\underset{\text{area}}{A(D)} \le \iint_D f(x,y)\,dA \le M\,A(D)$$

Proof:

$m \le f(x,y) \le M$ for all (x,y) in D (given)

$$\Rightarrow \iint_D m\,dA \le \iint_D f(x,y)\,dA \le \iint_D M\,dA \quad (\text{by } \#8)$$

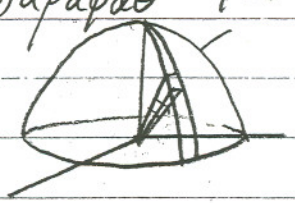
$$\iint_D m\,dA = m\iint_D dA \qquad \iint_D M\,dA = M\iint_D dA \quad (\text{by } \#7)$$

$$m\,\underset{\text{area}}{A(D)} \qquad M\,\underset{\text{area}}{A(D)} \quad (\text{by } \#10)$$

so $m(A) \le \iint_D f(x,y)\,dA \le M(A)$.

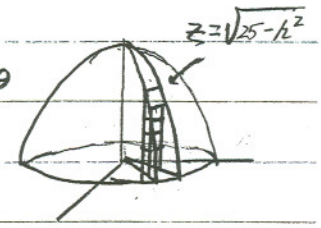
10. spherical coords:

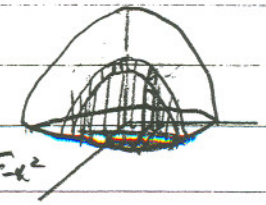
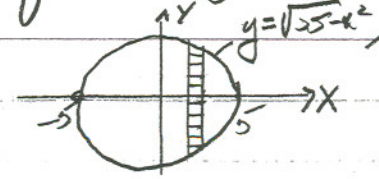
(a) $$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{5} f(\rho,\theta,\phi)\,\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$$



cylindrical Coords.

(b) $$\int_{\theta=0}^{2\pi}\int_{r=0}^{5}\int_{z=0}^{\sqrt{25-r^2}} f(r,\theta,z)\,r\,dz\,dr\,d\theta$$



rectangular coords:

(c) $$\int_{x=-5}^{5}\int_{y=-\sqrt{25-x^2}}^{\sqrt{25-x^2}}\int_{0}^{\sqrt{25-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$$