CALCULUS III **FINAL EXAMINATION**

(SM221/SM221P)

0755-1055, WEDNESDAY, 13 DECEMBER, 2017

NAME: SOLUTIONS **ALPHA NUMBER:** ______________

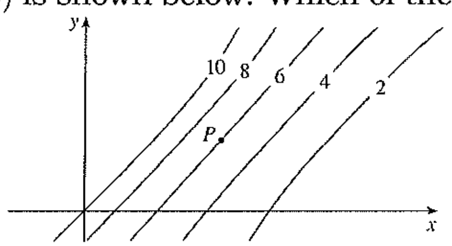
INSTRUCTOR: ______________ **SECTION:** ______________

Write your name, alpha number, section number and instructor's name on this page and on your bubble sheet. Bubble in your alpha number in the space provided on the left side of your bubble sheet.

PART ONE. Multiple Choice (50%). The first 20 problems are multiple choice. Fill in the letter for the best answer on the bubble sheet. There is no additional penalty for a wrong answer. You may use your TI-36X Pro calculator. The transformations from cartesian to cylindrical and spherical coordinates are included at the bottom of the last page of this test.

1. A contour map for a function $f(x, y)$ is shown below. Which of the following is true at the point P?

(a) $f_x(P) > 0$ and $f_y(P) > 0$
(b) $f_x(P) > 0$ and $f_y(P) < 0$
(c) $f_x(P) < 0$ and $f_y(P) > 0$
(d) $f_x(P) < 0$ and $f_y(P) < 0$
(e) $f_x(P) = 1$ and $f_y(P) = 0$



At P, f decreases in the direction of the positive x-axis; it increases in the direction of the positive y-axis

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2. Let $f(x, y)$ be the function from problem 1. Which of the following vectors approximates best the direction of $\nabla f(P)$?

(a) $\langle 1, 0\rangle$
(b) $\langle 0, 1\rangle$
(c) $\langle 1, -1\rangle$
(d) $\langle -1, 1\rangle$
(e) $\nabla f(P)$ is the zero vector

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3. The directional derivative of the function $g(x, y, z) = z^3 - x^2y$ at the point $(1, 6, 2)$ in the direction of the vector $\langle 3, 4, 12\rangle$ is equal to

(a) 13
(b) 41
(c) 104
(d) 0
(e) 8

$\nabla g(x,y,z) = \langle -2xy, -x^2, 3z^2\rangle$; $\nabla g(1,6,2) = \langle -12, -1, 12\rangle$

$\vec{u} = \langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13}\rangle$

$D_{\vec{u}}g(1,6,2) = \langle -12, -1, 12\rangle \cdot \langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13}\rangle$

$= \frac{-36-4+144}{13} = \frac{104}{13} = 8$

4. If $T(x, y) = 2x^2 + xy + y^2$ is the temperature at the point (x, y) on the plane, in which direction should one move from the point $(1, 2)$ in order for the temperature to increase the fastest?

(a) $\langle 1, 2\rangle$
(b) $\langle 2, 1\rangle$
(c) $\langle 4, 4\rangle$
(d) $\langle 5, 6\rangle$
(e) $\langle 6, 5\rangle$

$\nabla T(x,y) = <4x+y, x+2y>$

$\nabla T(1,2) = <6,5>$

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5. If $z = f(x, y)$ and $x = g(t)$, $y = h(t)$ where f, g, and h are differentiable functions such that

$$g(3) = 2, \quad g'(3) = 5, \quad h(3) = -1, \quad h'(3) = 2$$

$$f_x(2, -1) = 4, \quad f_y(2, -1) = -2$$

then $\frac{dz}{dt}$ when $t = 3$ is equal to

(a) 44
(b) 16
(c) 10
(d) 8
(e) 2

$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$. When $t=3$, $x=2$, $y=-1$

So $\left.\frac{dz}{dt}\right|_{t=3} = 4\cdot 5 + (-2)\cdot 2 = 16$

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6. If the velocity of a particle at time t is $\vec{v}(t) = \langle 2\sin t, \cos t\rangle$ and its position at time $t = 0$ is $\vec{r}(0) = \langle 1, -1\rangle$ then its position $\vec{r}(\pi)$ at time $t = \pi$ is

(a) $\langle 2, 1\rangle$
(b) $\langle 3, 1\rangle$
(c) $\langle 3, -1\rangle$
(d) $\langle 5, 1\rangle$
(e) $\langle 5, -1\rangle$

$\vec{r}(t) = <-2\cos t, \sin t> + \vec{c}$

$\vec{r}(0) = <-2, 0> + \vec{c} := <1,-1> \leadsto \vec{c} = <3,-1>$

So $\vec{r}(t) = <-2\cos t + 3, \sin t - 1>$

$\vec{r}(\pi) = <5, -1>$

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7. A tangent vector to the curve $\mathbf{r}(t) = \langle t + 1, t^2, t^2\rangle$ at the point $(0, 1, 1)$ is

(a) $\langle 1, 2t, 2t\rangle$
(b) $\langle 1, -2, -2\rangle$
(c) $\langle 1, 0, 0\rangle$
(d) $\langle 1, 2, 2\rangle$
(e) $\langle 0, 2, 2\rangle$

$\vec{r}(t) = <0,1,1>$ when $t=-1$

$\vec{r}'(t) = <1, 2t, 2t>$

$\vec{r}'(-1) = <1,-2,-2>$

8. The length of the parabolic arc $x = y^2$ from $(0,0)$ to $(4,2)$ is equal to

(a) $\sqrt{20}$

(b) $\int_0^4 \sqrt{t^2 + t^4}\, dt$

(c) $\int_0^2 \sqrt{1 + 4t^2}\, dt$

(d) $\int_0^4 \int_0^2 \sqrt{x^2 + y^4}\, dy\, dx$

(e) $\int_0^4 \int_0^2 \sqrt{4x^2 + y}\, dy\, dx$

$\vec{r}(t) = \langle t^2, t \rangle \quad 0 \leq t \leq 2$

$\vec{r}\,'(t) = \langle 2t, 1 \rangle \; ; \; \|\vec{r}\,'(t)\| = \sqrt{1+4t^2}$

$\ell = \int_0^2 \sqrt{1+4t^2}\, dt$

9. The tangent plane to the surface $z = \underbrace{4 - x^2 + y}_{f(x,y)}$ at the point $(1, 2, 5)$ has equation

(a) $-2x + y + z = 5$

(b) $z = 5 + 2x(x-1) + (y-2)$

(c) $2x - y + z = 0$

(d) $-2x + y + z = 0$

(e) $2x - y + z = 5$

$f(1,2) = 5$

$f_x(x,y) = -2x \; ; \; f_x(1,2) = -2$

$f_y(x,y) = 1 \; ; \; f_y(1,2) = 1$

Eqn of tangent plane: $z = 5 - 2(x-1) + 1(y-2)$

$\rightarrow z = -2x + y + 5 \rightarrow 2x - y + z = 5$

10. Let $f(t,p)$ be the speed of sound, in meters per second, when the temperature is t degrees Celsius and the pressure is p atmospheres. The statement $\frac{\partial f}{\partial t}(2, 10) = 0.4$ means

(a) When the temperature is 2 degrees Celsius and the pressure 10 atmospheres, the speed of sound is 0.4 meters per second.

(b) When the temperature is 2 degrees Celsius and the pressure 10 atmospheres, the speed of sound increases by 40%.

(c) When the temperature is 2 degrees Celsius and the pressure 10 atmospheres, then for every degree of increase in temperature the speed of sound increases by approximately 0.4 meters per second.

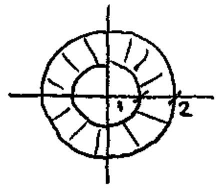
(d) When the temperature is 2 degrees Celsius and the pressure 10 atmospheres, the speed of sound increases by approximately 0.4 meters per second.

(e) When the temperature is 2 degrees Celsius and the pressure 10 atmospheres, the maximum rate of increase of the speed of sound is 0.4 meters per second.

11. Let D be the region on the plane between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 1$. Which of the following is equal to $\iint_D 1\, dA$?

(I) $\int_0^{2\pi} \int_1^2 r\, dr\, d\theta$ (II) $2\int_{-2}^{2} \left(\sqrt{4 - x^2} - \sqrt{1 - x^2}\right)\, dx$ (III) $\iint_D (2x - 2y)\, dA$

(a) (I) only
(b) (II) only
(c) (I) and (II) only
(d) (I) and (III) only
(e) (I), (II) and (III)



$0 \le \theta \le 2\pi$
$1 \le r \le 2$

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12. The volume of the solid region inside the sphere $x^2 + y^2 + z^2 = 4$, underneath the cone $z = \sqrt{x^2 + y^2}$ and above the xy-plane is given by

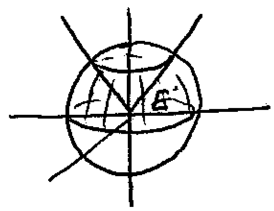
(a) $\displaystyle\int_0^{2\pi} \int_0^2 \int_0^r r\, dz\, dr\, d\theta$

(b) $\displaystyle\int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^2 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

(c) $\displaystyle\int_0^{2\pi} \int_0^2 \int_r^{\sqrt{4-r^2}} 1\, dz\, dr\, d\theta$

(d) $\displaystyle\int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^2 1\, d\rho\, d\phi\, d\theta$

(e) $\displaystyle\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(\sqrt{x^2 + y^2} - \sqrt{4 - x^2 - y^2}\right)\, dx\, dy$



$0 \le \theta \le 2\pi$
$\pi/4 \le \varphi \le \pi/2$
$0 \le \rho \le 2$

$V = \iiint_E 1\, dV$
$= \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^2 \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$

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13. Reversing the order of integration, the integral $\int_0^1 \int_{x^2}^{\sqrt{x}} xy^2\, dy\, dx$ becomes

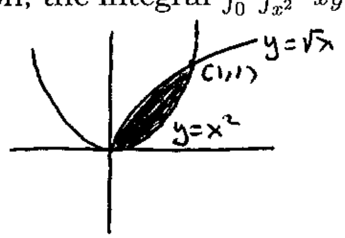
(a) $\displaystyle\int_0^1 \int_{\sqrt{y}}^{y^2} xy^2\, dx\, dy$

(b) $\displaystyle\int_0^1 \int_{y^2}^{\sqrt{y}} xy^2\, dx\, dy$

(c) $\displaystyle\int_0^1 \int_{\sqrt{y}}^{y^2} yx^2\, dx\, dy$

(d) $\displaystyle\int_{x^2}^{\sqrt{x}} \int_0^1 xy^2\, dx\, dy$

(e) $\displaystyle\int_{\sqrt{x}}^{x^2} \int_0^1 xy^2\, dx\, dy$



$0 \le y \le 1$
$y^2 \le x \le \sqrt{y}$

14. Which of the following is a parametrization for the cone $y^2 = x^2 + z^2$?
(a) $\mathbf{r}(u, v) = \langle \cos v, u, \sin v \rangle$
(b) $\mathbf{r}(u, v) = \langle \cos v, v, \sin v \rangle$
(c) $\mathbf{r}(u, v) = \langle u \cos v, 1, u \sin v \rangle$
(d) $\mathbf{r}(u, v) = \langle u \cos v, u, u \sin v \rangle$ $(u\cos v)^2 + (u \sin v)^2 = u^2$
(e) $\mathbf{r}(u, v) = \langle \cos v, 1, \sin v \rangle$

15. The work done by the force field $\mathbf{F}(x, y) = \langle y, x \rangle$ along the straight line segment from the point $(1, 1)$ to the point $(2, 4)$ is equal to
(a) 0 $\vec{F}$ is conservative, with potential function $f(x,y) = x \cdot y$
(b) 7
(c) 13 $W = f(2,4) - f(1,1) = 8 - 1 = 7$
(d) 14
(e) 28

16. For which value of the constant k is the vector field $\mathbf{F}(x, y) = \langle \underbrace{y^2 + ky \sin(2x)}_{P}, \underbrace{2xy + \cos(2x)}_{Q} \rangle$ conservative?
(a) -2 $\frac{\partial Q}{\partial x} = 2y - 2\sin(2x)$
(b) -1
(c) 0 $\frac{\partial P}{\partial y} = 2y + k \sin(2x)$
(d) 1
(e) 2 We must have $k = -2$ for $\vec{F}$ to be conservative

17. Suppose that $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ is a vector field whose components have continuous partial derivatives in $\mathbb{R}^2$. Let D be the disk $x^2 + y^2 \leq 1$ and C its boundary. If it is known that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$, which of the following must be true?

(I) $\mathbf{F}$ is conservative
(II) $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$
(III) $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = 0.$ True by Green's thm.

(a) (I) only
(b) (II) only
(c) (III) only
(d) (I), (II) and (III)
(e) None of the three statements is necessarily true

18. The outward flux $\iint_S \mathbf{F} \cdot d\mathbf{S}$ of the vector field $\mathbf{F}(x, y, z) = \langle 3x - 4xz, y^2 + xz, 2z^2 - 2yz \rangle$ through the surface S which is the boundary of the solid cube $0 \leq x \leq 2$, $0 \leq y \leq 2$, $0 \leq z \leq 2$ is

(a) 24
(b) 0
(c) 3
(d) 8
(e) -8

By the divergence thm, $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV$ (where E is the solid cube)

$\iiint_E \operatorname{div} \vec{F}\, dV = \iiint_E 3 - 4z + 2y + 4z - 2y\, dV = 3 \iiint_E 1\, dV$

$= 3 \cdot (\text{volume of } E) = 24$

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19. Suppose that $\mathbf{F}$ is a vector field whose components have continuous partial derivatives in $\mathbb{R}^3$. Let S be the sphere $x^2 + y^2 + z^2 = 1$ with the outward orientation, S_1 the top hemisphere with the upward orientation and S_2 the bottom hemisphere with the downward orientation. If it is known that $\operatorname{div}\mathbf{F} = 0$, which of the following statements must be true?

(I) $\mathbf{F}$ is conservative.
(II) $\iint_S \mathbf{F} \cdot d\mathbf{S} = 0$. True by divergence thm
(III) $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$.

(a) (I) only
(b) (II) only
(c) (I) and (II) only
(d) (I) and (III) only
(e) (I), (II) and (III)

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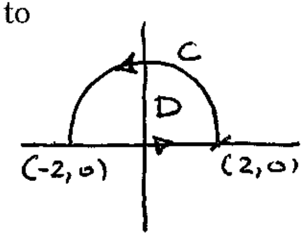
20. If C is the curve consisting of the semicircle $x^2 + y^2 = 1$, $y \geq 0$ from $(2, 0)$ to $(-2, 0)$ followed by the straight line segment from $(-2, 0)$ to $(2, 0)$, then the line integral $\int_C (x^2 - y)\, dx + (x + y^3)\, dy$ is equal to

(a) -8π
(b) -4π
(c) 0
(d) 4π
(e) 8π



By Green's thm

$\int_C (x^2 - y)\, dx + (x + y^3)\, dy =$

$= \iint_D 1 - (-1)\, dA = 2 \iint_D 1\, dA$

$= 2 \cdot \frac{1}{2} 4\pi = 4\pi$

END OF MULTIPLE CHOICE PORTION OF THE TEST

PART TWO. Free response (50%). The remaining 10 problems are not multiple choice. Each is worth 10 points. Answer them in the space provided on these pages. Show all your work. You may use your TI-36X Pro calculator. No other aides are allowed.

21. (10 points) The table below shows values of the function $f(x, y)$, defined on the rectangle $R = [0, 8] \times [0, 20]$.

(a) Estimate the values of $f_x(4, 10)$ and $f_y(4, 10)$.

$y \setminus x$	0	2	4	6	8
0	20	19	13	11	8
5	23	20	14	13	11
10	28	25	20	17	14
15	33	31	24	19	14
20	41	40	30	26	20

$$f_x(4,10) \approx \frac{f(6,10)-f(2,10)}{4} = \frac{17-25}{4} = -2$$

$$f_y(4,10) \approx \frac{f(4,15)-f(4,5)}{10} = \frac{24-14}{10} = 1$$

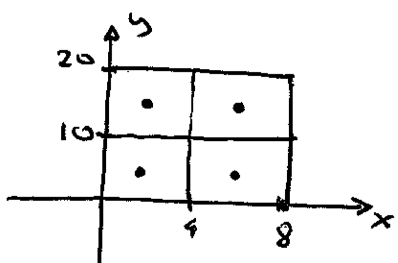
(b) Use linear approximation to estimate $f(4.2, 9.9)$.

$$L(x,y) = 20 - 2(x-4) + 1\cdot(y-10)$$

$$f(4.2, 9.9) \approx L(4.2, 9.9) = 20 - 2(0.2) + (-0.1)$$

$$= 19.5$$

(c) Estimate $\iint_R f(x, y)\, dA$ using the midpoint rule with four sub-rectangles of equal size.



$\Delta A = 40$

$$\iint_R f(x,y)\, dA \approx \left(f(2,5)+f(2,15)+f(6,5)+f(6,15)\right)\Delta A$$

$$= (20+31+13+19)\cdot 40$$

$$= 83\cdot 40 = 3320$$

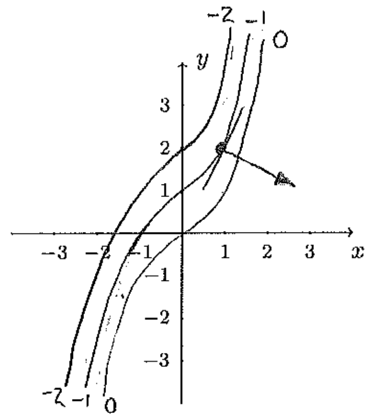
22. (10 points) Consider the function $f(x,y) = x^3 - y$.

(a) Sketch a contour map of f that includes the level curves at levels $z = 0, -1$ and -2.

$z=0$: $x^3-y=0 \rightsquigarrow y=x^3$

$z=-1$: $x^3-y=-1 \rightsquigarrow y=x^3+1$

$z=-2$: $x^3-y=-2 \rightsquigarrow y=x^3+2$



(b) On the contour map from part (a) and at the point $(1, 2)$, sketch accurately the direction in which f increases at the fastest possible rate.

(c) What is the maximum rate of increase of f at the point $(1, 2)$?

$\nabla f(x,y) = \langle 3x^2 - 1 \rangle$

$\nabla f(1,2) = \langle 3, -1 \rangle$

max rate of increase of f at $(1,2)$: $\|\nabla f(1,2)\| = \sqrt{10}$

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23. (10 points) Find the critical points of the function $f(x,y) = xy^2 - x + 2y$ and determine whether they are points of local maximum, local minimum or saddle points.

$\left.\begin{array}{l} f_x = y^2 - 1 = 0 \\ f_y = 2xy + 2 = 0 \end{array}\right\} \rightsquigarrow \begin{array}{cc} y=1 & \text{or } y=-1 \\ \downarrow & \downarrow \\ x=-1 & x=1 \end{array}$

Critical points $(-1,1)$ and $(1,-1)$

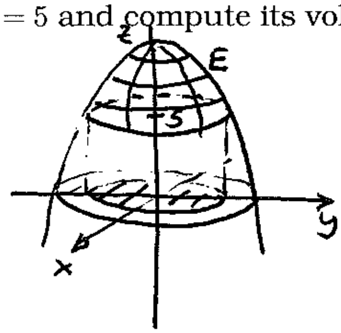
$f_{xx} = 0$, $f_{yy} = 2x$, $f_{xy} = f_{yx} = 2y$

$D(x,y) = 0 \cdot 2x - 4y^2 = -4y^2$

$D(-1,1) = -4 < 0$ so $(-1,1)$ is a saddle point

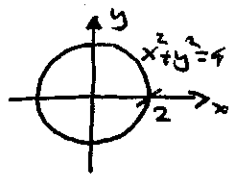
$D(1,-1) = -4 < 0$ so $(1,-1)$ is a saddle point.

24. (10 points) Sketch the region E bounded by the paraboloid $z = 9 - x^2 - y^2$ and the plane $z = 5$ and compute its volume.



Intersection: $9 - x^2 - y^2 = 5$

$\Rightarrow x^2 + y^2 = 4$



E: $0 \le \theta \le 2\pi$
$0 \le r \le 2$
$5 \le z \le 9 - r^2$

$$V = \iiint_E 1\, dV = \int_0^{2\pi} \int_0^2 \int_5^{9-r^2} r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^2 \left(rz \Big|_{z=5}^{9-r^2} \right) dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^2 4r - r^3\, dr\, d\theta$$

$$= \int_0^{2\pi} \left(2r^2 - \frac{r^4}{4} \Big|_{r=0}^{2} \right) d\theta$$

$$= \int_0^{2\pi} 4\, d\theta$$

$$= 8\pi$$

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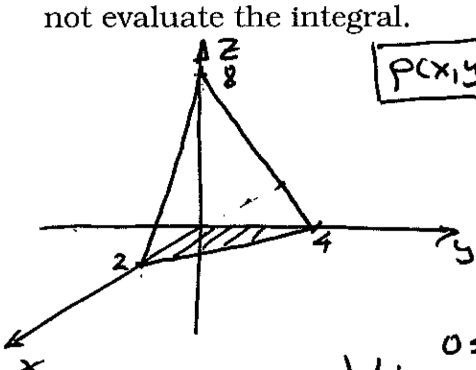
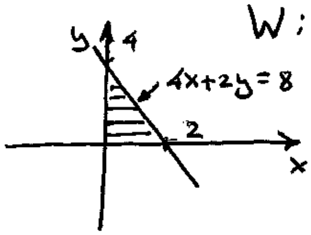
25. (10 points) Let W be the solid region in the first octant bounded by the plane $4x+2y+z = 8$ and the three coordinate planes. If the density at the point (x, y, z) is numerically equal to the distance from the xy-plane, set up a triple iterated integral equal to the mass of W. Do not evaluate the integral.



$\boxed{\rho(x,y,z) = z}$



W: $0 \le x \le 2$
$0 \le y \le 4 - 2x$
$0 \le z \le 8 - 4x - 2y$

$$m = \iiint_W z\, dV$$

$$= \int_0^2 \int_0^{4-2x} \int_0^{8-4x-2y} z\, dz\, dy\, dx$$

(other answers also possible)

26. (10 points) Consider the vector field $\mathbf{F}(x, y, z) = \langle 2xy + z + 3x^2, x^2 + 2yz, x + y^2 \rangle$.

(a) Show that F is conservative and find a potential function.

$\vec{F}$ is everywhere defined and $\operatorname{curl}\vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy+z+3x^2 & x^2+2yz & x+y^2 \end{vmatrix} = \langle 2y-2y, -(1-1), 2x-2x \rangle$

$= \langle 0,0,0 \rangle$

Hence $\vec{F}$ is conservative.

Potential function: $f(x,y,z) = x^2y + xz + x^3 + y^2 z$

(b) Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve $\mathbf{r}(t) = \langle \cos t, \sin t, 2 - \sin t \rangle$, $0 \le t \le \pi$.

$\int_C \vec{F} \cdot d\vec{r} = f(-1,0,2) - f(1,0,2) = (-2-1) - (2+1)$

$= -6$

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27. (10 points) Compute the flux $\iint_S \mathbf{F} \cdot d\mathbf{S}$ of the vector field $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$ through the cylinder $x^2 + y^2 = 4$, $0 \le z \le 3$ with the outward orientation (i.e., the normal vector points away from the z-axis).

S: $\vec{r}(\theta,z) = \langle 2\cos\theta, 2\sin\theta, z \rangle$. D: $0 \le \theta \le 2\pi$, $0 \le z \le 3$

$\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial z} = \begin{vmatrix} i & j & k \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 2\cos\theta, 2\sin\theta, 0 \rangle$

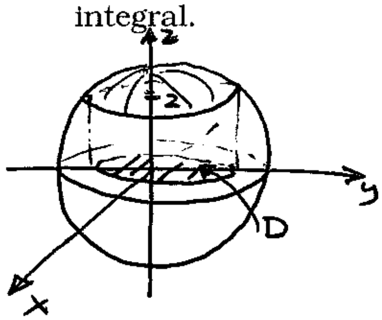
"outward" orientation

$$\iint_S F \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(\theta,z)) \cdot \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial z}\, dA = \iint_D \langle 2\cos\theta, 2\sin\theta, z \rangle \cdot \langle 2\cos\theta, 2\sin\theta, 0 \rangle\, dA$$

$$= \iint_D 4\, dA = 4 \cdot (\text{area of } D) = 4 \cdot 6\pi = 24\pi$$

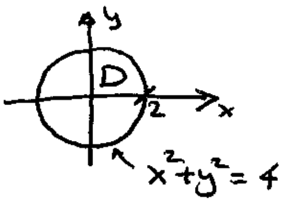
Other solutions using the divergence theorem to change the surface are possible.

28. (10 points) Let S be the part of the sphere $x^2 + y^2 + z^2 = 8$ that lies above the plane $z = 2$. Set up a double iterated integral that is equal to the surface area of S. Do not evaluate the integral.



S: $\vec{r}(x,y) = \langle x, y, \sqrt{8-x^2-y^2} \rangle$

(x,y) in D

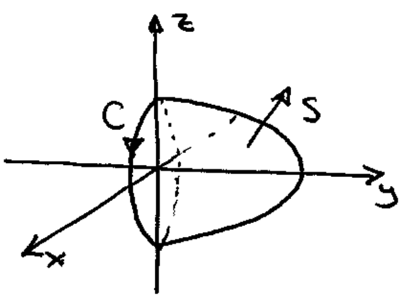


$$\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \begin{vmatrix} i & j & k \\ 1 & 0 & \frac{-x}{\sqrt{8-x^2-y^2}} \\ 0 & 1 & \frac{-y}{\sqrt{8-x^2-y^2}} \end{vmatrix} = \left\langle \frac{x}{\sqrt{8-x^2-y^2}}, \frac{y}{\sqrt{8-x^2-y^2}}, 1 \right\rangle$$

$$\left\| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right\| = \sqrt{\frac{x^2}{8-x^2-y^2} + \frac{y^2}{8-x^2-y^2} + 1} = \frac{\sqrt{8}}{\sqrt{8-x^2-y^2}}$$

$$\text{Area of } S = \iint_D \frac{\sqrt{8}}{\sqrt{8-x^2-y^2}}\, dA = \int_0^{2\pi} \int_0^2 \frac{\sqrt{8}}{\sqrt{8-r^2}} \cdot r \, dr \, d\theta$$

29. (10 points) Compute the surface integral $\iint_S \text{curlF} \cdot d\text{S}$ where $\text{F}(x, y, z) = \langle z + y, x^2 z, -x + z^2 y \rangle$ and S is the paraboloid $y = 1 - x^2 - z^2$, $y \geq 0$ with the outward orientation (i.e., the normal vector has positive y component).



By Stokes' thm, $\iint_S \text{curl}\, \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$

where C is a circle of radius 1 on the xz-plane (centered at the origin) with clockwise orientation

C: $\vec{r}(t) = \langle \cos t, 0, -\sin t \rangle \quad 0 \leq t \leq 2\pi$

So, $\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt =$

$$= \int_0^{2\pi} \langle -\sin t, -\cos^2 t \sin t, -\cos t \rangle \cdot \langle -\sin t, 0, -\cos t \rangle dt$$

$$= \int_0^{2\pi} \sin^2 t + \cos^2 t \, dt$$

$$= 2\pi$$

Conclusion: $\iint_S \text{curl}\, \vec{F} \cdot d\vec{S} = 2\pi$

30. (10 points) Prove the following Theorem: Let C be a smooth curve given by the vector function $\mathrm{r}(t)$, $a \le t \le b$. Let f be a differentiable function of three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathrm{r} = f(\mathrm{r}(b)) - f(\mathrm{r}(a))$$

Pf: $\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$, by definition. If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

we have: $\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) = \frac{\partial f}{\partial x}(\vec{r}(t)) \cdot x'(t) + \frac{\partial f}{\partial y}(\vec{r}(t)) \cdot y'(t) + \frac{\partial f}{\partial z}(\vec{r}(t)) \cdot z'(t)$

which, by the chain rule, is equal to $\frac{d}{dt}(f(\vec{r}(t)))$.

So $\int_C \nabla f \cdot d\vec{r} = \int_a^b \frac{d}{dt}(f(\vec{r}(t))) dt = f(\vec{r}(t))\Big|_{t=a}^{b}$ (by the fundamental theorem of calculus)

$= f(\vec{r}(b)) - f(\vec{r}(a))$

Spherical Coordinates

$x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \sin\theta$
$z = \rho \cos\phi$

Cylindrical Coordinates

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$