SM221 Practice Test IV Solutions (Melles)

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Part I – No calculators allowed.

1. Derive the formula for the element of area $r\,dr\,d\theta$ for double integrals in polar coordinates.

We can derive this formula using the method of 16.7, Equation 2 on p. 1111. Let S be the xy-plane, parametrized as

$\boldsymbol{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 0\rangle$, where $r \geq 0$. The element of area for S is given by $dS = |\boldsymbol{r}_r \times \boldsymbol{r}_\theta|\,dr\,d\theta$. Calculate $\boldsymbol{r}_r$ and $\boldsymbol{r}_\theta$ and show that $|\boldsymbol{r}_r \times \boldsymbol{r}_\theta| = r$.

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Part II –Calculators are allowed for this part of the test.

2. Identify the surface given by each parametrization.

a. $\vec{r}(x,y) = \langle x, y, 3 - x - y \rangle$ Plane

b. $\vec{r}(u,v) = \langle u, v, 3u^2 + 3v^2 \rangle$ Paraboloid $z = 3x^2 + 3y^2$

c. $\vec{r}(\theta,z) = \langle 3\cos\theta, 3\sin\theta, z \rangle$ Cylinder of radius 3, centered on the z-axis

d. $\vec{r}(\varphi,\theta) = \langle 3\sin\varphi\cos\theta, 3\sin\varphi\sin\theta, 3\cos\varphi \rangle$ Sphere of radius 3 centered at the origin

e. $\vec{r}(r,\theta) = \langle 3r\cos\theta, 3r\sin\theta, 3r \rangle$ Cone $x^2 + y^2 = z^2$

3. Use the Divergence Theorem to find the flux $\iint_S \boldsymbol{F} \cdot \boldsymbol{dS}$, where S is the surface of the solid bounded by the cylinder $y^2 + z^2 = 4$ and the planes $x = 0$ and $x = 3$, with the outward orientation, and $\boldsymbol{F}(x,y,z) = y\boldsymbol{i} + x\boldsymbol{j} + 5z\boldsymbol{k}$.

Answer: 60π

4. Use Stokes' Theorem to find $\iint_S \operatorname{curl} \boldsymbol{F} \cdot \boldsymbol{dS}$, where $\boldsymbol{F}(x,y,z) = 3y\boldsymbol{i} + 5x\boldsymbol{j} + 4y\boldsymbol{k}$ and S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$, with the upward orientation. Be sure to show clearly how Stokes' Theorem was used.

Answer: 8π

5. Evaluate the surface integral $\iint_S \boldsymbol{F} \cdot \boldsymbol{dS}$, where $\boldsymbol{F}(x,y,z) = 4x\boldsymbol{i} + z\boldsymbol{j} + 5\boldsymbol{k}$ and S is the cylinder given by the parametrization $\boldsymbol{r}(u,v) = \langle 2\cos v, u, 2\sin v \rangle$, for $0 \leq u \leq 3$ and $0 \leq v \leq 2\pi$ with the outward orientation.

Answer: 48π

6. Evaluate the surface integral $\iint_S \sqrt{1 + 4y}\, dS$, where S is the part of the paraboloid $y = x^2 + z^2$ such that $x^2 + z^2 \leq 1$.

Answer: 3π