The exam totals 200 points, 10 for each long answer question and 5 for each multiple choice question. You can use a plebe issue calculator in any capacity. No other aids are allowed and no collaboration is permitted. THESE ARE SOLUTIONS TO THE EXAM.

Answer questions 1-20 on the bubble sheets provided.

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1. After a u-substitution, the definite integral $\int_1^2 x(x^2+1)^5\,dx$ equals which of the following?

(A) $\int_1^2 u^5\,du$ (B) $\int_2^5 u^5\,du$ (C) $\frac{1}{2}\int_2^5 u^5\,du$ (D) $\frac{1}{12}u^6$ (E) $\frac{1}{2}\int_1^2 u^5\,du$

Set $u = x^2+1$ so $du = 2x\,dx$ and $\frac{1}{2}du = x\,dx$. When $x=1$, $u=2$ and when $x=2$, $u=5$. So the integral becomes $\dfrac{1}{2}\displaystyle\int_2^5 u^5\,du$. The answer is C.

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2. If dV/dt represents the rate (in acres/minute) of rainforest lost per minute to deforestation, what does the integral $\displaystyle\int_0^{60} \frac{dV}{dt}\,dt$ represent?

(A) The number of trees lost per minute.
(B) The volume of trees lost in an hour.
(C) The number of acres of rainforest lost in an hour.
(D) The number of acres of rainforest lost in a minute.
(E) None of the other answers.

The integral represents the net loss of rainforest (in acres) in 60 minutes (i.e. 1 hour). The answer is C.

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3. Consider the value $A = \displaystyle\int_0^1 1-x^2\,dx$. Let L_n represent the left-endpoint Riemann sum estimate of the integral with n boxes and R_n represent the right-endpoint Riemann sum estimate of the integral with n boxes. Which of the following inequalities is correct?

(A) $R_3 < R_6 < A < L_6 < L_3$ (B) $A < L_3 < L_6 < R_6 < R_3$ (C) $L_6 < L_3 < A < R_3 < R_6$
(D) $R_6 < R_3 < A < L_3 < L_6$ (E) $R_3 < L_3 < A < R_6 < L_6$

Drawing boxes helps us see that $R_3 < R_6 < A < L_6 < L_3$. Since the function is decreasing on the interval of integration the right endpoint estimates are below the value of the integral and the left endpoint estimates are above the value of the integral. Moreover, using more boxes should bring us closer to the correct value for the integral. This reasoning supports the answer that we could deduce using graphical means. The answer is A.

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4. After one integration by parts, the indefinite integral $\int e^x\cos(x)\,dx$ could equal which one of the following?

(A) $-\int e^x\sin(x)\,dx$ (B) $\int -e^x\sin(x)+e^x\cos(x)\,dx$ (C) $e^x\cos(x)+\int e^x\sin(x)\,dx$
(D) $e^x\sin(x)+C$ (E) $e^x\cos(x)-\int e^x\cos(x)\,dx$

Set $u=\cos(x)$ and $dv = e^x\,dx$. Then $du = -\sin(x)\,dx$ and $v = e^x$. The integral becomes $uv - \int v\,du = e^x\cos(x)+\int e^x\sin(x)\,dx$. The answer is C.

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5. Use Euler's method, $y_{\text{new}} = y_{\text{old}} + (slope)\Delta x$, with two steps of step size $\Delta x = 3$ to estimate the value of y when $x = 6$ if y satisfies the initial value problem $\frac{dy}{dx} = x - y$, $y(0) = 1$. The estimate for $y(6)$ is closest to:

(A) 6 (B) 12 (C) 18 (D) 24 (E) 30

We make a small table:

x	y	$y' = x - y$	$\Delta y = y' * \Delta x = 3y'$
0	1	-1	-3
3	-2	5	15
6	13		

Euler's method predicts that when $x = 6$, $y = 13$. The answer is B.

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6. Which of the following is a solution to the separable differential equation $\frac{dy}{dx} = \frac{y^2}{x}$?

(A) $y = \ln \mid x \mid + C$ (B) $y = 1$ (C) $x = 0$ (D) $y = Ce^{\frac{\ln|x|}{2}}$ (E) $y = \frac{-1}{\ln|x|+C}$

Rearrange the differential equation to get $\frac{dy}{y^2} = \frac{dx}{x}$. Integrate both sides to get $-\frac{1}{y} = \ln \mid x \mid + C$, so $y = \frac{-1}{\ln \mid x \mid + C}$. The answer is E.

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7. What is the limit of the sequence $a_n = \frac{3n + 7n^3 - 4}{n^3 + 6n^2 + \ln(n) + 2}$ as $n \to \infty$?

(A) 3 (B) 0 (C) ∞ (D) 7 (E) Cannot be determined

Dividing all the terms on the top and bottom of the fraction by n^3 and letting $n \to \infty$ we get

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\frac{3}{n^2} + 7 - \frac{4}{n^3}}{1 + \frac{6}{n} + \frac{\ln(n)}{n^3} + \frac{2}{n^3}} = 7.$$

Here we used the fact that n^3 grows much faster than $\ln(n)$ so that $\frac{\ln(n)}{n^3} \to 0$ as $n \to \infty$. The answer is D.

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8. Which of the following series converge?

(I) $\sum_{n=1}^{\infty} \frac{1}{n}$

(II) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

(III) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

(A) I and II only (B) II and III only (C) I only (D) III only (E) I, II and III

Series (I) is the harmonic series and diverges (e.g. by the p-test with $p = 1$). Series (II) is an alternating series such that the absolute values of the terms decrease to zero, so the series is convergent by the Alternating Series Test. Series (III) is a p-series with $p = 2$ and so it converges by the p-test (since $p > 1$). The answer is B.

9. Which of the following conditions guarantees that the series $\sum_{n=0}^{\infty} a_n$ converges?

(A) $\lim_{n\to\infty} a_n = 0$.
(B) a_n is decreasing to zero.
(C) The sequence of partial sums $a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots$ converges.
(D) The sequence of partial sums is bounded.
(E) The terms a_n alternate between positive and negative values.

To say that a series converges is precisely to assert that its sequence of partial sums converges, so the answer is C. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is a counterexample to (A) and (B), while the alternating series $\sum_{n=1}^{\infty}(-1)^n$ is a counterexample to (D) and (E).

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10. Suppose a power series, centered at $x = 3$, converges at $x = 7$ and diverges at $x = -3$. The power series must:

(A) have radius of convergence equal to ∞ (B) diverge at $x = 10$ (C) converge at $x = -7$
(D) have radius of convergence equal to 4 (E) converge at $x = 9$

Since the power series converges at $x = 7$ and has center $x = 3$, we know that the radius of convergence must be at least 4. As well, since the power series diverges at $x = -3$ and has center $x = 3$, the radius of convergence can't be larger than 6. It follows that the answer is B: the power series must diverge at $x = 10$, which is 7 units away from the center $x = 3$, which is larger than the radius of convergence. Answer (A) is impossible since the radius of convergence is no larger than 6. Answer (C) is impossible since the distance from $x = -7$ to the center is 10 and this is larger than the radius of convergence. The radius of convergence could be equal to 4 but this is not necessary so answer (D) is false. It is possible for the power series to converge at $x = 9$ (which is 6 units away from the center and so $x = 9$ would be right on the boundary of the open interval of convergence if the radius of convergence was 6), but this is not necessary so answer (E) is false.

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11. Which of the following is a degree-3 Taylor polynomial for $f(x) = x^3 + 3x^2 - 2x + 1$ centered at 1?

(A) $(x-1)^3 + 3(x-1)^2 - 2(x-1) + 1$
(B) $x^3 + 3x^2 - 2x + 1$
(C) $6(x-1)^3 + 12(x-1)^2 + 7(x-1) + 3$
(D) $(x-1)^3 + 6(x-1)^2 + 7(x-1) + 3$
(E) $6(x+1)^3 + 12(x+1)^2 + 7(x+1) + 3$

Computing derivatives gives $f^{(0)}(1) = f(1) = 3$, $f^{(1)}(1) = 7$, $f^{(2)}(1) = 12$ and $f^{(3)}(1) = 6$. Since the degree-3 Taylor polynomial for $f(x)$ centered at 1 is $\sum_{n=0}^{3} \frac{f^{(n)}(1)}{n!}(x-1)^n$, we find that the Taylor polynomial is

$$3 + 7(x-1) + \frac{12}{2!}(x-1)^2 + \frac{6}{3!}(x-1)^3.$$

Simplifying the coefficients and writing the polynomial in order of decreasing powers of $(x-1)$ gives answer D.

12. The distance between $e^{3\pi i/2}$ and -1 is

(A) 0 (B) 1 (C) $\sqrt{2}$ (D) 2 (E) 4

The value $e^{3\pi i/2}$ equals $-i$ and so is plotted on the complex plane at $(0,-1)$. The point -1 is plotted on the complex plane at $(-1,0)$. The distance between these two points is $\sqrt{(0-(-1))^2+(-1-0)^2} = \sqrt{2}$ so the answer is C.

13. The polar coordinates for the point $(x,y) = (-1,-1)$ are:

(A) $r = 2$ and $\theta = \pi/4$ (B) $r = -2$ and $\theta = \pi/4$ (C) $r = 2$ and $\theta = 5\pi/4$
(D) $r = \sqrt{2}$ and $\theta = \pi/4$ (E) $r = \sqrt{2}$ and $\theta = 5\pi/4$

The point has $r = \sqrt{(-1)^2+(-1)^2} = \sqrt{2}$ and angle $\theta = \pi + \arctan(-1/-1) = 5\pi/4$ (we need to add π to the arctan value since the point is to the left of the y-axis – recall that arctan is a multi-valued function). So the answer is E.

14. Given vectors $\vec{a} = \langle 1,2,3\rangle$ and $\vec{b} = \langle -1,0,1\rangle$, we have:

(A) $\vec{a}\bullet\vec{b} = \langle -1,0,3\rangle$ and $\vec{a}\times\vec{b} = \langle -2,4,-2\rangle$ (B) $\vec{a}\bullet\vec{b} = 2$ and $\vec{a}\times\vec{b} = \langle -2,4,-2\rangle$
(C) $\vec{a}\bullet\vec{b} = 2$ and $\vec{a}\times\vec{b} = \langle 2,-4,2\rangle$ (D) $\vec{a}\bullet\vec{b} = -2$ and $\vec{a}\times\vec{b} = \langle -2,4,-2\rangle$
(E) $\vec{a}\bullet\vec{b} = 2$ and $\vec{a}\times\vec{b} = \langle 2,4,2\rangle$

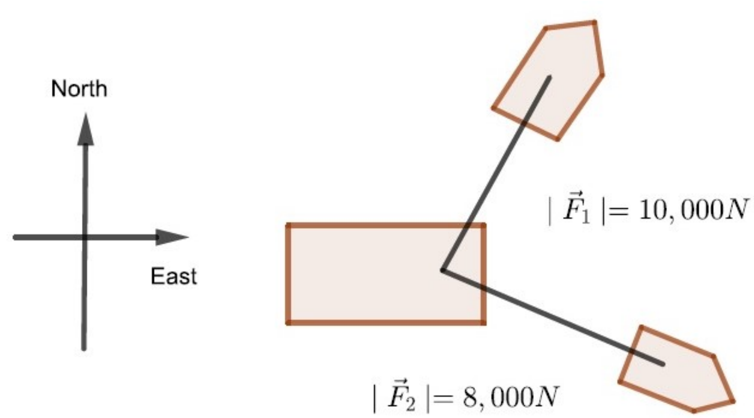
We have $\vec{a}\bullet\vec{b} = 2$ and $\vec{a}\times\vec{b} = \langle 2,-4,2\rangle$ so the answer is C.

15. A vector equation for the line through the points A(1,0,0) and B(2,3,-1) is

(A) $\vec{r}(t) = \langle -1+t,3,-1\rangle$ (B) $\vec{r}(t) = \langle 1+2t,3t,-t\rangle$ (C) $\vec{r}(t) = \langle 1+t,3t,-t\rangle$
(D) $\vec{r}(t) = \langle 1,t,3t\rangle$ (E) $\vec{r}(t) = \langle 1+3t,3t,-t\rangle$

A vector pointing along the line is $\vec{AB} = \langle 1,3,-1\rangle$ and we can use A as the starting point of our parameterization, so the parametric equations could be $x = 1+1t$, $y = 0+3t$, and $z = 0-1t$. Converting these into vector equation form gives the correct answer, C.

16. Two tug boats pull a supply raft to shore. The first pulls with a force of 10,000N at an angle 60° North of East and the second pulls with a force of 8,000N at an angle 30° South of East, as pictured below.



The resultant (net) force is closest to:

(A) 18,000N at 30° North of East (B) 2,000N at 30° North of East (C) 12,000N East
(D) 12,800N at 20° North of East (E) 14,000N at -20° North of East

Break the two forces into components: $\vec{F_1} = \langle 10000\cos(60°), 10000\sin(60°)\rangle = \langle 5000, 5000\sqrt{3}\rangle$ and $\vec{F_2} = \langle 8000\cos(-30°), 8000\sin(-30°)\rangle = \langle 4000\sqrt{3}, -4000\rangle$. Then the resultant vector is $\vec{F} = \vec{F_1}+\vec{F_2} = \langle 5000+4000\sqrt{3}, 5000\sqrt{3}-4000\rangle \approx \langle 11928, 4660\rangle$. Then $|\,\vec{F}\,| \approx \sqrt{11928^2+4660^2} \approx 12806N$ and $\theta = \arctan(4660/11928) \approx 0.372$ rad $\approx 21.3°$. The answer is D.

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17. A line through $P(1,0,1)$ that is perpendicular to the plane $x+2y+z=12$ hits the plane $x+y-z=6$ at the point:

(A) $(6,0,0)$ (B) $(4,6,4)$ (C) $(0,6,0)$ (D) $(3,0,-3)$ (E) $(6,5,-4)$

A vector perpendicular to the plane $x+2y+z=12$ is $\langle 1,2,1\rangle$. So we can use $x=1+1t$, $y=0+2t$, and $z=1+1t$ as parametric equations for the line. A particle moving along this line hits the plane $x+y-z=6$ when $(1+t)+(0+2t)-(1+t)=6$, i.e. when $t=3$. This corresponds to the point with $x=1+1(3)=4$, $y=0+2(3)=6$ and $z=1+1(3)=4$ on the line. So the point of intersection of the line and the given plane is $(4,6,4)$ and the correct answer is B.

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18. The improper integral $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ equals

(A) -1 (B) 0 (C) 1 (D) ∞ (E) $\sum_{n=1}^\infty \frac{1}{n^2}$

We have $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \lim_{T\to\infty}\int_1^T \frac{1}{x^2}\,dx = \lim_{T\to\infty}\left(\frac{-1}{T}+\frac{1}{1}\right) = 1$. So the answer is C.

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19. Which of the following approximates the sum of the convergent alternating series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n^2}$ accurate to within $\frac{1}{30}$ using the fewest number of terms as given by the Alternating Series Estimation Theorem?

(A) $\frac{1}{2}-\frac{1}{8}$ (B) $\frac{1}{2}-\frac{1}{8}+\frac{1}{18}$ (C) $\frac{1}{2}-\frac{1}{8}+\frac{1}{18}-\frac{1}{32}$
(D) $\frac{1}{2}-\frac{1}{8}+\frac{1}{18}-\frac{1}{32}+\frac{1}{50}$ (E) $\frac{1}{2}-\frac{1}{8}+\frac{1}{18}-\frac{1}{32}+\frac{1}{50}-\frac{1}{72}$

The Alternating Series Estimation Theorem says that if we want to approximate the value of a convergent alternating series to within error ϵ we can omit all terms whose absolute value is less than ϵ. It follows that to approximate this series to within $1/30$ we can omit the fourth term $\frac{-1}{32}$ and all subsequent terms. So the answer is B.

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20. When using partial fractions to evaluate the integral $\displaystyle\int \frac{1}{(x)^2(x+1)(x^2+1)}\,dx$ you should first rewrite the integrand as a sum of fractions of the form

(A) $\frac{A}{x}+\frac{B}{x}+\frac{C}{x+1}+\frac{D}{x^2+1}$ (B) $\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}+\frac{D}{x^2+1}$ (C) $\frac{A}{x}+\frac{B}{x^2}+\frac{C}{(x+1)^3}+\frac{D}{(x^2+1)^4}$
(D) $\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}+\frac{Dx}{x^2+1}$ (E) $\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}+\frac{Dx+E}{x^2+1}$

The form of the numerator over an irreducible quadratic like x^2+1 is the sum of a linear and constant term $Dx+E$, so the only possible answer is (E) and this is the correct answer.

Complete questions 21-30, in blue books, or on paper, as directed by your instructor.

21. (a) Compute the indefinite integral $\int x^{2022} \ln(x)\, dx$.

(b) Compute the definite integral $\int_1^c \frac{1}{x(x+1)}\, dx$, where c is a constant greater than 1. Show all your work for full credit.

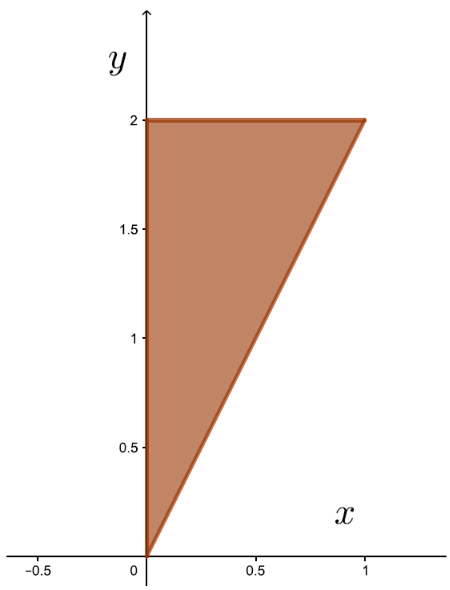
For part (a) use integration by parts with $u = \ln(x)$ and $dv = x^{2022}\, dx$. The resulting expression can be evaluated to give $\frac{x^{2023}}{2023} \ln(x) - \frac{x^{2023}}{2023^2} + C$.

For part (b), first use partial fractions to show that

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}.$$

Then the integral equals $(\ln(x) - \ln(x+1))\,|_1^c$, which evaluates to $\ln\left(\frac{2c}{c+1}\right)$.

22. Rotate the region (pictured below) in the first quadrant that lies above the line $y = 2x$, below the line $y = 2$ and to the right of the y-axis about the y-axis to produce a solid of revolution.



(a) Set up an integral to compute the volume of revolution using the disk or washer method. You need not evaluate your integral but if you want to do so to check your answer, the volume is $2\pi/3$ cubic units.

(b) Now set up an integral to compute the volume of revolution using the shell method.

For part (a) we get the integral $\int_0^2 \pi \left(\frac{y}{2}\right)^2\, dy$. For part (b) we get the integral $\int_0^1 2\pi(x)(2 - 2x)\, dx$. Both integrals evaluate to $2\pi/3$.

23. Recall Hooke's Law: the force required to move a spring is proportional to the distance the spring has been stretched. A spring is held 10cm past its natural length by a force of 5N. Find the work required to stretch the spring from 10cm past its natural length to 30cm past its natural length.

First calculate the spring constant using $F = kx$: $5N = k(0.1m)$ so $k = 50N/m$. Now compute the work using an integral

$$W = \int_{0.1}^{0.3} (50x)\, dx = 2J.$$

24. Fifty feet of heavy rope hangs from the rooftop over the side of a building that is 100 feet tall. The rope weighs 4 lbs/ft. Find the work done in pulling the rope to the top of the building.

The integral can be set up in one of two ways. We can slice the rope so that a piece of rope of length dy is y feet from the top of the building. The work required to raise this rope to the top of the building is $Fd = (4dy\ lb)(y\ ft)$ and we integrate this quantity to find the total work done: $W = \int_0^{50} 4y\, dy = 5000\, ft \cdot lbs$. Another way to get the answer is to imagine that $y\, ft$ of rope has already been lifted to the roof. Then the remaining rope weighs $4(50 - y)\, lb$ and the work done to raise this rope $dy\, ft$ is $W = Fd = 4(50 - y)\, dy$. Integrate this quantity to find the total work done in raising the rope to the roof: $W = \int_0^{50} 4(50 - y)\, dy = 5000\, ft \cdot lbs$.

25. (a) Explain why the series $\sum_{n=1}^{\infty} \frac{4}{3^n} = (4/3) + (4/9) + \cdots$ converges and find the value to which it converges.

(b) Does the series $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ converge or diverge? Give reasons to support your answer.

The series in part (a) is a geometric series with $a = 4/3$ and $r = 1/3$. The series converges because $| r |= 1/3 < 1$ and it converges to $\frac{a}{1-r} = \frac{4/3}{1-(1/3)} = 2$. There are many ways to do part (b). Since $0 < \frac{1}{n^2+1} < \frac{1}{n^2}$ and since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (by the p-test with $p = 2 > 1$), we see that the series $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ converges by the comparison test. Another way to see that the series converges is to use the integral test. Since

$$\int_1^{\infty} \frac{1}{x^2+1}\, dx = \lim_{T\to\infty} \int_1^{T} \frac{1}{x^2+1}\, dx = \lim_{T\to\infty} \arctan(T) - \arctan(1) = \frac{\pi}{2} - \frac{\pi}{4} < \infty,$$

the integral converges and so does the series.

26. (a) Find a vector perpendicular to the plane that contains the three points $A(3,0,0)$, $B(0,5,0)$ and $C(0,0,4)$.
(b) Find an equation for the plane from part (a).
(c) Find the area of the triangle ABC.

For (a), $\vec{AB} = \langle -3,5,0\rangle$ and $\vec{AC} = \langle -3,0,4\rangle$, so $\vec{n} = \vec{AB}\times\vec{AC} = \langle 20,12,15\rangle$ is a vector perpendicular to the plane. For (b), the equation of the plane is $20x+12y+15z = d$ and since point A must satisfy this equation, $d = 20(3)+12(0)+15(0) = 60$. So the equation of the plane is $20x+12y+15z = 60$. For (c), the area of the triangle is

$$\frac{|\,\vec{AB}\times\vec{AC}\,|}{2} = \frac{|\,\langle 20,12,15\rangle\,|}{2} = \frac{\sqrt{20^2+12^2+15^2}}{2} = \frac{\sqrt{769}}{2} \approx 13.865.$$

27. A 40 foot by 30 foot rectangular swimming pool has depth 8 feet but is only filled by 6000 cubic feet of water.
(a) How deep is the water?
(b) How much work is done to pump all the water to the top of the pool? Use $\delta = 62.5$ lbs/ft^3 as the weight density of water.

For (a), the water has volume $40\,ft\times 30\,ft\times d\,ft = 1200d\,ft^3$, where d is the depth of the water. This equals $6000\,ft^3$ so $d = 5\,ft$. For (b), we slice the water at a constant height $y\,ft$. The slice has thickness $dy\,ft$. The volume of the slice is $(40)(30)(dy)\,ft^3$ and this can be converted into a weight by multiplying by δ. The weight is $(62.5)(40)(30)(dy)\,lb$. This slice must be lifted $8-y\,ft$ and this requires $(62.5)(40)(30)(8-y)(dy)\,ft\cdot lbs$ of work. Integrating this quantity computes the total work done:

$$W_{\text{total}} = \int_0^5 (62.5)(40)(30)(8-y)\,dy = 2,062,500\,ft\cdot lbs.$$

Here we integrated from height $y=0$ (at the bottom of the pool) to height $y=5$ (at the surface of the water).

28. (a) Professor Lipton loves instant iced tea made by mixing powder with water. He starts to make a big batch by filling a tank with 100L of water and adding 1kg of powder. After tasting the mixture, he decides that it would be better to add 0.3L/min of more concentrated iced tea containing 0.02kg of powder per liter to the tank. His sneaky son drinks 0.3L/min from the tank though, removing some of the powdered mixture. Let A(t) represent the amount (in kg) of powder in the tank at time t minutes. Write an initial value problem governing A(t), **but do not solve it**.

(b) Find the limiting value of A(t) (as t goes to infinity). Hint: this can be done without solving the initial value problem.

For part (a) we are given that $A(0) = 1$. This is the initial value condition. The equation $\frac{dA}{dt} = \text{rate of } A \text{ in} - \text{rate of } A \text{ out}$ gives the differential equation condition

$$\frac{dA}{dt} = (0.3)(0.02) - \left(\frac{A}{100}\right)(0.3) = 0.006 - 0.003A.$$

For (b) you can set $\frac{dA}{dt}$ equal to 0 and solve to get $A = 2$. You can also realize that the concentration of the powder in the tank is going to approach the concentration of the powder in the fluid added to the tank, so the limiting value of $A(t)$ will be $(0.02\,kg/L)(100\,L) = 2\,kg$.

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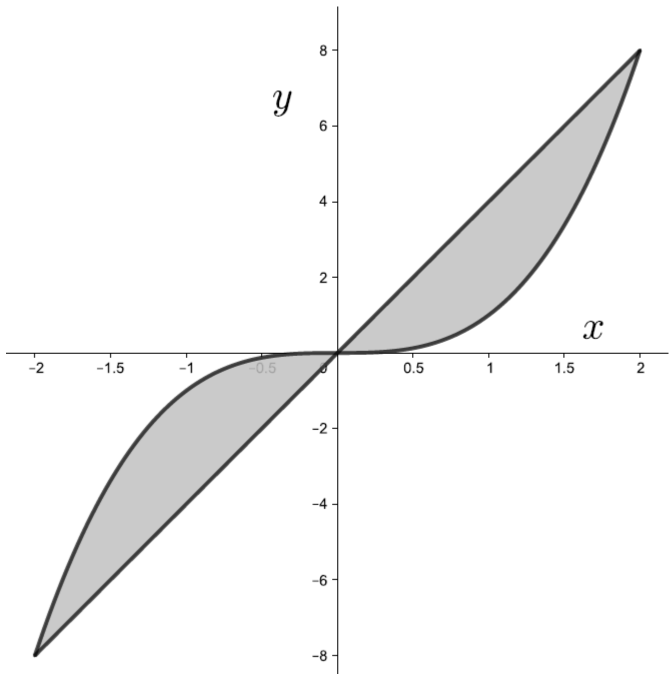
29. Find the area of the shaded region enclosed between the curves $y = x^3$ and $y = 4x$, as pictured below.



The area equals

$$\int_{-2}^{0} x^3 - 4x\,dx + \int_{0}^{2} 4x - x^3\,dx = 2\int_{0}^{2} 4x - x^3\,dx = 8 \text{ units}^2.$$

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30. Read the following example of how to compute a "line integral" (a new kind of integral that you'll explore in Calculus III) and then compute a line integral for a second example as described below.

Example: A force field $\vec{F}(x,y) = \langle x+y, 2+y\rangle$ assigns a force vector $\langle x+y, 2+y\rangle$ to each point (x,y) in the plane (suppose force is measured in Newtons and distance in meters). A particle is moving through the plane along a path parameterized by the vector equation $\vec{r}(t) = \langle x(t), y(t)\rangle = \langle 1+t, 3-t\rangle$ for $0 \le t \le 1$. At time t, the particle has velocity given by $\vec{r}\,'(t) = \langle x'(t), y'(t)\rangle = \langle 1, -1\rangle$ and so over a small time interval of length Δt its displacement is $\langle 1, -1\rangle\Delta t$. The work done by the force field as the particle is displaced is approximately

$$\begin{aligned} W_{\text{interval}} &= \vec{F}(\vec{r}(t)) \bullet \vec{r}\,'(t)\Delta t \\ &= \vec{F}(x(t), y(t)) \bullet \vec{r}\,'(t)\Delta t \\ &= \langle x(t) + y(t), 2 + y(t)\rangle \bullet \langle 1, -1\rangle\Delta t \\ &= \langle (1+t) + (3-t), 2 + (3-t)\rangle \bullet \langle 1, -1\rangle\Delta t \\ &= [((1+t) + (3-t)) * 1 + (2 + (3-t)) * (-1)]\Delta t \\ &= (t-1)\Delta t. \end{aligned}$$

Breaking the path into equal-length intervals, adding up the estimates of work done over each interval, and taking a limit as the number of intervals goes to infinity, we obtain an integral that gives the work done by the force field $\vec{F}$ along the path:

$$W_{\text{total}} = \int_0^1 (t-1)dt = -0.5N \cdot m = -0.5J.$$

Your turn: Using the above example as a guide, show that the work done by the force field $\vec{F}(x,y) = \langle -y, x\rangle$ as a particle moves once around the circle of radius three, following a parameterized path $\vec{r}(t) = \langle 3\cos(t), 3\sin(t)\rangle$ for $0 \le t \le 2\pi$, is equal to 18π Joules.

$$\begin{aligned} W_{\text{interval}} &= \vec{F}(3\cos(t), 3\sin(t)) \cdot \langle -3\sin(t), 3\cos(t)\rangle\, dt \\ &= \langle -3\sin(t), 3\cos(t)\rangle \cdot \langle -3\sin(t), 3\cos(t)\rangle\, dt \\ &= (9\sin^2(t) + 9\cos^2(t))\, dt \\ &= 9\, dt \end{aligned}$$

so $W_{\text{total}} = \int_0^{2\pi} 9\, dt = 18\pi\ J.$

Well done! You've reached the end of the exam. But do go back and check your answers. Often you can find a creative way to check that your answer is correct or think of another argument that leads to the same conclusion, giving you more confidence in your answers.
