NAME: SOLUTIONS ALPHA NUMBER: _____
INSTRUCTOR: _____ SECTION: _____

CALCULUS I (SM121, SM121A, SM131) FINAL EXAMINATION

0755-1055 Monday 15 December 2008 SHOW ALL WORK ON THIS TEST PACKAGE

Tear off the last sheet of this exam. It contains problems 27, 28, 29, 30. Do these problems first WITHOUT YOUR CALCULATOR. When you are finished with these 4 problems, return the sheet to your instructor, take out you calculator, and complete the rest of the exam.

PART ONE: MULTIPLE CHOICE (50%). The first 20 problems are multiple-choice. Fill in the best answer on your Scantron bubble sheet. Write your name, alpha number, instructor and section on this test and your bubble sheet and bubble in your alpha number. There is no extra penalty for wrong answers on the multiple choice. Show all your scratch work on this test.

CALCULATORS ARE PERMITTED FOR THIS MULTIPLE CHOICE SECTION.

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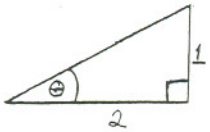
1. For the right triangle shown, which trigonometric function applied to θ gives a result of $\frac{\sqrt{5}}{2}$?



a) sin b) cos c) tan d) cot e) sec

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2. For the triangle in problem #1, the angle θ (in radians) is closest to which of the following?

a) 0.46 b) 0.50 c) $\pi/6$ d) $\pi/3$ e) 2.00

$\tan^{-1}(1/2) = 0.46$

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3. If $\log_a(x) = 2$ and $\log_a(y) = 3$, find $\log_a(x^2/y)$. $= 2\log_a(x) - \log_a(y) = 2(2) - 3 = 1$

a) 0 b) 1 c) 2 d) 3 e) 4

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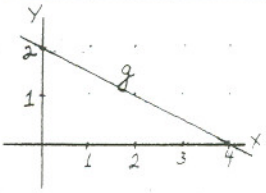
4. If $f(x) = x^2 + 1$ and g^{-1} is the inverse for the function g graphed on the right, find $f(g^{-1}(1))$.

$= f(2) = 2^2 + 1 = 5$



a) 1 b) 2 c) 3 d) 4 e) 5

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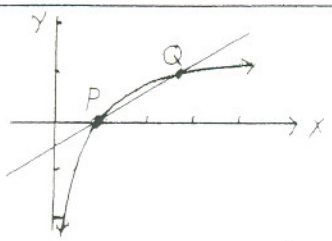
5. The graph for $y = \ln(x)$ is sketched on the right. Find the slope of the line through points P and Q. (P has an x coordinate of 1 and Q has an x coordinate of e.)



a) $\frac{1}{2}$ b) 1 c) $\frac{1}{e-1}$

d) e e) $e-1$

$$m = \frac{\Delta y}{\Delta x} = \frac{\ln(e) - \ln(1)}{e-1} = \frac{1-0}{e-1}$$

6. Find $\lim_{x \to 1^+} f(x)$ where $f(x) = \begin{cases} x^2, & x \le 1 \\ \sin(\pi x), & 1 < x \end{cases}$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \sin(\pi x)$
$= \sin(\pi) = 0$

a) -1 b) 0 c) 1
d) $\pi/2$ e) none

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7. If f is a function with $f(1) = 3$ and $f'(1) = 2$, then the tangent line to the graph of $y = f(x)$ at the point corresponding to $x = 1$ has a y intercept equal to :

$y - y_0 = m(x - x_0)$
$y - 3 = 2(x-1) = 2x - 2$
$y = 2x - 2 + 3 = 2x + 1$

a) -1 b) 0 c) $2/3$
d) 1 e) $\pi/2$

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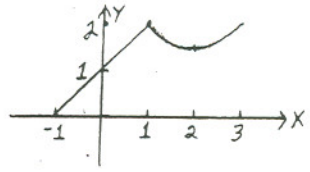
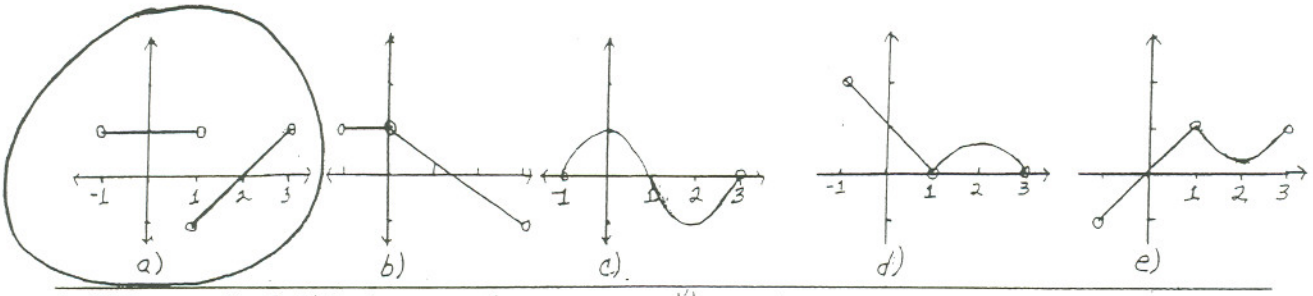
8. If f is a function whose graph is shown on the right, which of the following could be the graph for f' ?





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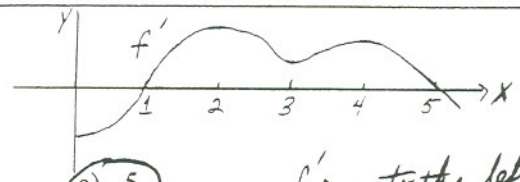
9. If the graph of f' is shown on the right, then the graph of f has a local maximum at what x value?



a) 1 b) 2 c) 3 d) 4 e) 5

$f' > 0$ to the left of 5
$f' < 0$ to the right of 5

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10. If the graph of f' is shown in problem #9, then f'' is maximum at:

a) 1 b) 2 c) 3 d) 4 e) 5

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11. Find the slope of the line tangent to the curve given implicitly by $x^2 y - y^3 = 3$ at the point (2 , 1).

$\frac{d}{dx}\left[x^2 y - y^3\right] = \frac{d}{dx}[3]$

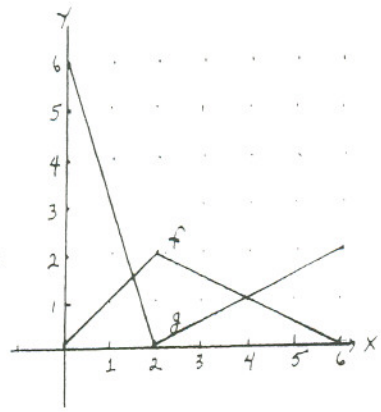
$2xy + x^2 \frac{dy}{dx} - 3y^2 \frac{dy}{dx} = 0$

$\left[x^2 - 3y^2\right] \frac{dy}{dx} = -2xy$

$\frac{dy}{dx} = \frac{-2xy}{x^2 - 3y^2}\Big|_{(2,1)} = \frac{-2(2)(1)}{2^2 - 3(1)^2} = \frac{-4}{4-3} = -4$

a) 1
b) -1
c) 4
d) -4
e) none of the above

Use the graphs of the functions f and g on the right to solve problems #12 and #13.



12. If $h(x) = f(x)/g(x)$, find $h'(1) =$ $\dfrac{f'(1)\,g(1) - f(1)\,g'(1)}{(g(1))^2}$

a) -3 b) 0 c) 2/3
d) 6 e) d.n.e.

$= \dfrac{(1)(3) - (1)(-3)}{3^2} = \dfrac{6}{9} = \dfrac{2}{3}$

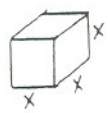
13. If $k(x) = f(g(x))$, find $k'(1) =$ $f'(g(1)) \cdot g'(1)$

$= f'(3) \cdot g'(1)$

$= (-\tfrac{1}{2})(-3) = \tfrac{3}{2}$

a) -3 b) 1 c) 4/3
d) 3/2 e) d.n.e.

14. Find the rate of change of the volume of a cube with respect to time at the moment when the edge length (x) of the cube is $2m$ and is increasing at a rate of $0.1\ m/s$.



$V = x^3$

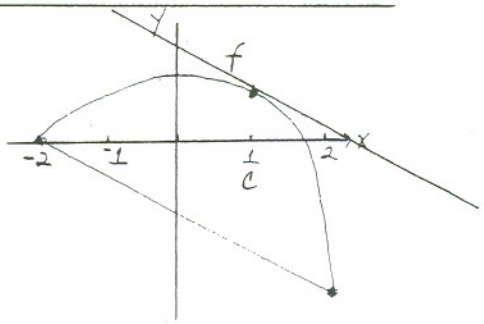
$\dfrac{dV}{dt} = 3x^2 \dfrac{dx}{dt}$

$\dfrac{dV}{dt} = 3(2m)^2(.1\ m/s)$

$= 1.2\ m^3/s$

a) $0.1\ m^3/s$ b) $8\ m^2/s$ c) $0.8\ m^3/s$
d) $1.2\ m^3/s$ e) $0.8\ m/s$

15. The Mean Value Theorem states that for the function f graphed on the right there is at least one number c in the interval $[a, b] = [-2, 2]$ where $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. Based on the graph, which of the following numbers is the best extimate for c ?



a) -2 b) -1 c) 0
d) 1 e) 2

16. Use your calculator to sketch the graphs for $y = e^{-x}$ and $y = x^3 + x^2 - 1$ and determine the x – coordinate of the point of intersection accurate to two decimal places.

a) -0.20 b) 0.87 c) 1.15 d) 2.72 e) d.n.e.

17. Find $\frac{d}{dx}\left(\int_0^x t^2 \sin(t)\,dt\right)$.

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

a) $\cos(x^2)$ b) $x^2\sin(x)$ c) $\frac{x^3}{3}\sin(x)$ d) $\frac{x^3}{3}\cos(x)$ e) $-\frac{x^3}{3}\cos(x)$

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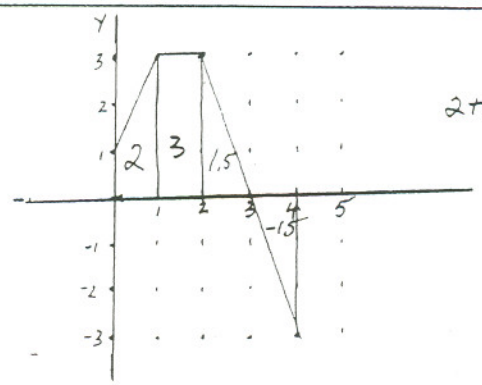
18. Determine $\int_0^4 f(x)dx$ for the function f whose graph on the right consists of 3 line segments.



$2+3+1.5-1.5=5$

a) 2 b) 3 c) 5
d) 6 e) 8

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19. If the table on the right gives the velocity of a car at various times, find the average acceleration (ft/s ^{2}) of the car over the time interval [1, 3].

t (sec)	0	1	2	3	4
v (ft/s)	0	5	15	31	51

$$\text{ave accel} = \frac{\Delta V}{\Delta t} = \frac{V(3)-V(1)}{3-1} = \frac{31-5}{2} = 13 \text{ ft/s}^2$$

a) 5 b) 13 c) 26 d) 31 e) 51

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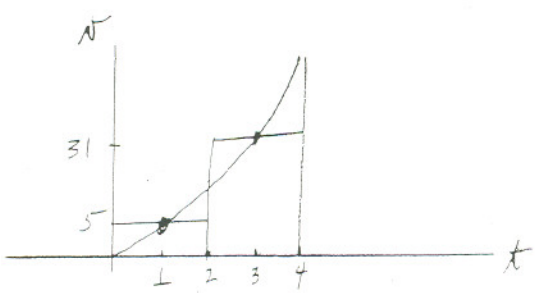
20. Use the table in problem #19 to approximate the distance traveled (ft) over the time interval [0 , 4] by using M_2, the midpoint method using two subintervals.

a) 13 b) 26 c) 51 d) 72 e) 102

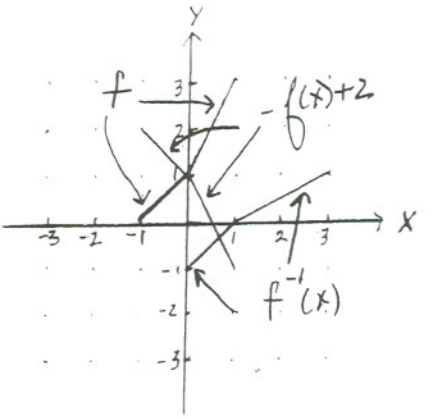
$$M_2 = V(1)\Delta t + V(3)\Delta t$$
$$= 5(2) + (31)(2)$$
$$= 10 + 62 = 72 \text{ ft}$$



$$\Delta t = \frac{4-0}{2} = 2$$

Part Two. Longer Answers (50%). These are not multiple choice. Again, SHOW ALL YOUR WORK ON THESE TEST PAGES. **CALCULATORS PERMITTED FOR ALL PROBLEMS <u>EXCEPT</u> 27, 28, 29, 30.**

21. The graph of a function f is shown on the right (two line segments).



a) Write the formula for $y = f(x)$ as a piece-wise defined function.

$$y = \begin{cases} x+1 ; & -1 \le x \le 0 \\ 2x+1 ; & 0 \le x \le 1 \end{cases}$$

b) Explain why f has an inverse function f^{-1}.

f is "one to one" (satisfies the horizontal line test) on $[-1,1]$.

c) Sketch the graph of f^{-1} on the same axes with f.

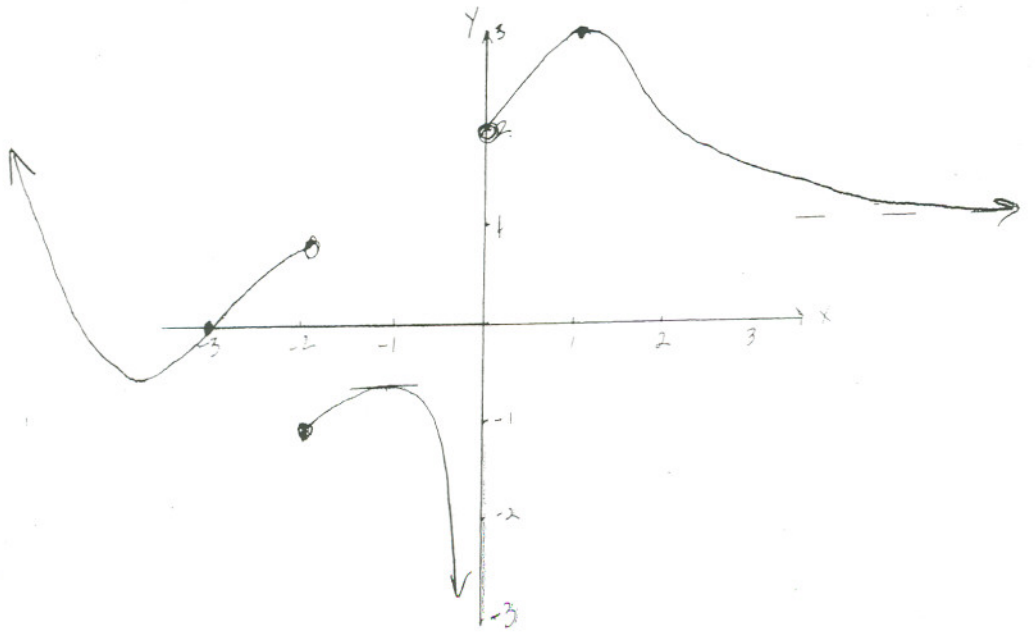
d) Sketch the graph of $y = -f(x) + 2$ on the same axes with f.

22. Sketch the graph <u>of a single function</u> (on the axes below) which satisfies all of the following:

a) $\lim_{x \to -3} f(x) = 0$; b) $\lim_{x \to -2^-} f(x) = 1$; c) $\lim_{x \to -2^+} f(x) = -1$;

d) $f'(-1) = 0$; e) $\lim_{x \to 0^-} f(x) = -\infty$; f) $\lim_{x \to 0^+} f(x) = 2$;

g) $\lim_{x \to 1} f(x) = 3$; h) $\lim_{x \to \infty} f(x) = 1$; i) $\lim_{x \to -\infty} f(x) = \infty$



23. a) Write the limit definition for $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

b) Use the limit definition to show that for $f(x) = x^2 + 3x$, $f'(x) = 2x + 3$.

$$f'(x) = \lim_{h \to 0} \frac{\left[(x+h)^2 + 3(x+h)\right] - \left[x^2 + 3x\right]}{h}$$

$$= \lim_{h \to 0} \left(\cancel{x^2} + 2xh + h^2 + \cancel{3x} + 3h \cancel{- x^2} \cancel{- 3x}\right)/h$$

$$= \lim_{h \to 0} \frac{\cancel{h}\left[2x + h + 3\right]}{\cancel{h}} = 2x + 3$$

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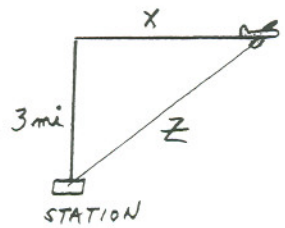
24. A plane flying horizontally at an altitude of 3 mi and a speed of 500 mi/hr passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when the plane is 5 mi from the station.



z = dist from plane to station

find $\frac{dz}{dt}$ when $z = 5$.

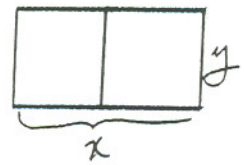
$$\frac{d}{dt}\left[x^2 + 3^2 = z^2\right]$$

$$\cancel{2}x\frac{dx}{dt} + 0 = \cancel{2}z\frac{dz}{dt}$$

$$\frac{dz}{dt} = \frac{x}{z}\frac{dx}{dt} = \frac{4\text{ mi}}{5\text{ mi}} 500\text{ mi/hr}$$ ($x = 4$ mi when $z = 5$ mi)

$$= 400\text{ mi/hr}$$

25. A farmer wants to enclose a rectangular area of 15,000 square feet, and then divide it into 2 pens with fencing parallel to one of the sides. What dimensions will minimize the cost of the fence?



minimize the perimeter $P = 2x + 3y$

where $xy = 15{,}000\ ft^2 \Rightarrow y = \frac{15{,}000}{x}$

$P = 2x + 3y = 2x + 3\left(\frac{15{,}000}{x}\right)$

$P(x) = 2x + \frac{45{,}000}{x}, \quad 0 < x.$

$P'(x) = 2 - 45{,}000/x^2 = 0$

$\Rightarrow 2 = \frac{45{,}000}{x^2}$

$\Rightarrow x^2 = 22{,}500$

$\Rightarrow x = 150.$

$P''(x) = 90{,}000/x^3 > 0$ when $0 < x$

$\Rightarrow x = 150$ ft, $y = 100$ ft give min.

26. A can of soda is taken from a refrigerator and placed on a table in a room of temperature 70° F at time t =0. Let $f(t)$ measure the temperature (° F) of the soda as a function of time t (mins). In all of the following questions, make sure that you give the units.

a) What does $f(0) = 40$ mean? At time $t=0$ (mins) the can has temp 40°F

b) What does $f(10) = 60$ mean? " $t=10$ " " 60°F

c) What is the average rate of change of the temperature over the time interval [0 , 10] ?

$$\frac{\Delta f}{\Delta t} = \frac{f(10) - f(0)}{10 - 0} = \frac{60 - 40}{10} = 2°F/\text{min}$$

d) What does $f'(10) = 0.5$ mean?

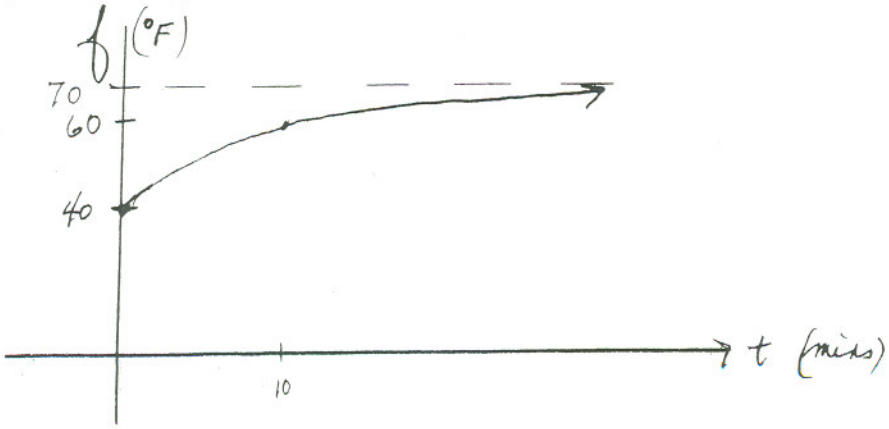
The instantaneous rate of change at $t = 10$ mins is 0.5°F/min.

e) Use the information in (b) and (d) to approximate $f(12)$.

$$f(12) \doteq f(10) + f'(10)(12-10) = 60 + (.5)(2) = 61°F$$

f) Find $\lim_{t\to\infty} f(t) = 70°F$ (room temp)

g) Sketch the graph for $y = f(t)$.



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CALCULATORS NOT PERMITTED FOR 27, 28, 29, 30.

27. a) Evaluate the following definite integral showing all of your steps:

$\int_{-1}^{2} (6t^2 + 6t + 6)\, dt$ $= \left[2t^3 + 3t^2 + 6t\right]\Big|_{-1}^{2} = \left[2(2)^3 + 3(2)^2 + 6(2)\right] - \left[2(-1)^3 + 3(-1)^2 + 6(-1)\right]$

$= (16 + 12 + 12) - (-2 + 3 - 6)$

$= 40 - (-5) = 45$

b) A particle moves vertically with acceleration $a(t) = 3e^t + \cos(t) + t$. Find its velocity $v(t)$ and position $s(t)$ if $v(0) = 3$ and $s(0) = 4$.

$v(t) = \int (3e^t + \cos(t) + t)\,dt = 3e^t + \sin(t) + t^2/2 + C$

$v(0) = 3 = 3\cancel{e^0}^{1} + \cancel{\sin(0)}^{0} + \cancel{0^2/2}^{0} + C \Rightarrow C = 0$

so $v(t) = 3e^t + \sin(t) + t^2/2$

$s(t) = \int (3e^t + \sin(t) + t^2/2)\,dt = 3e^t - \cos(t) + t^3/6 + C$

$s(0) = 4 = 3\cancel{e^0}^{1} - \cancel{\cos(0)}^{1} + \cancel{0^3/6}^{0} + C$

$4 = 2 + C \Rightarrow C = 2 \Rightarrow s(t) = 3e^t - \cos(t) + t^3/6 + 2$

28. Find $\frac{dy}{dx}$ for the following functions. (Do not simplify your answers.)

a) $y = (x-2)^3 \cos(5x)$; $\frac{dy}{dx} = 3(x-2)^2\cos(5x) + (x-2)^3(-\sin(5x))5$

b) $y = \frac{\sqrt[3]{x}}{e^{2x}} = \frac{x^{1/3}}{e^{2x}}$; $\frac{dy}{dx} = \left(\frac{1}{3}x^{-2/3}e^{2x} - x^{1/3}e^{2x}\cdot 2\right)/(e^{2x})^2$

c) $y = \tan(3x) + \tan^{-1}(3x)$; $\frac{dy}{dx} = \sec^2(3x)\cdot 3 + \frac{1}{1+(3x)^2}\cdot 3$

d) $y = x^{x^5}$; $\frac{d}{dx}\left[\ln(y) = x^5\ln(x)\right] \Rightarrow \frac{1}{y}y' = 5x^4\ln(x) + x^5\cdot\frac{1}{x}$

$\Rightarrow y' = x^{x^5}\left[5x^4\ln(x) + x^4\right]$

29. Use f' and f'' to graph $f(x) = x^4 - 2x^3$. Label all relative maximums and minimums and inflection points.

$f(x) = x^4 - 2x^3 = x^3(x-2)$

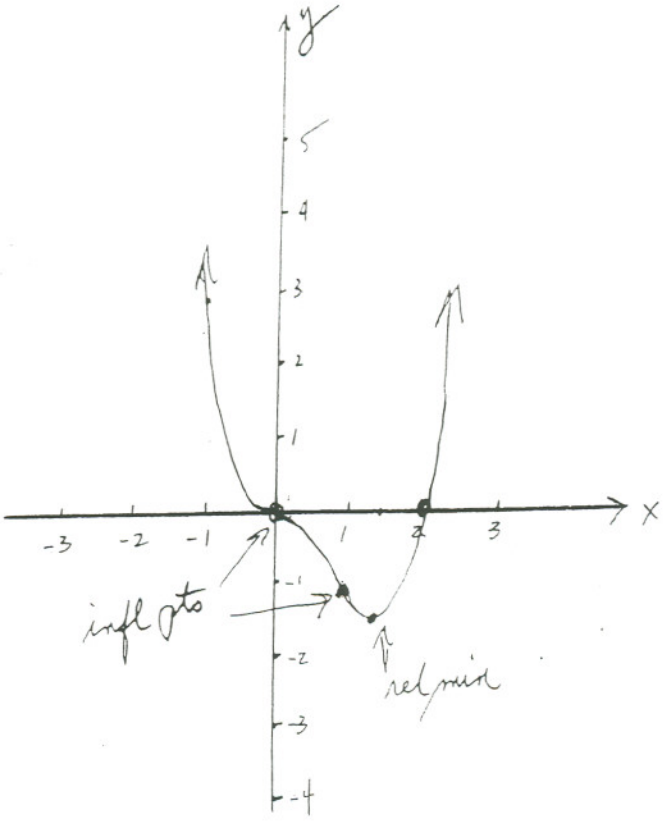
$f'(x) = 4x^3 - 6x^2 = 2x^2(2x-3)$

$f''(x) = 12x^2 - 12x = 12x(x-1)$

sign of f: $+ + + \; 0 \; - - - \; 0 \; + + +$ (0, 2)

sign of f': $- - - \; 0 \; - - - \; 0 \; + + +$ (0, 1.5, 2)

sign of f'': $+ + + \; 0 \; - - \; 0 \; + +$ (0, 1, 2)



30. Find the following limits. Show all work.

a) $\lim_{x\to 0} \frac{x^{10}+3}{2x^{10}+1} = \frac{3}{1} = 3$

b) $\lim_{x\to\infty} \frac{x^{10}+3}{2x^{10}+1} \overset{alg}{=} \lim_{x\to\infty} \frac{(x^{10}+3)/x^{10}}{(2x^{10}+1)/x^{10}} = \lim_{x\to\infty} \frac{1+3/x^{10}}{2+1/x^{10}} = 1/2$

c) $\lim_{x\to 0} \frac{\cos(x)-1}{x^2} \overset{L'}{=} \lim_{x\to 0} \frac{-\sin(x)-0}{2x} \overset{L'}{=} \lim_{x\to 0} -\frac{\cos(x)}{2} = -1/2$